

# Inserting Multiple Edges into a Planar Graph

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## Abstract

Let  $G$  be a connected planar (but not yet embedded) graph and  $F$  a set of additional edges not yet in  $G$ . The *multiple edge insertion* problem (MEI) asks for a drawing of  $G + F$  with the minimum number of pairwise edge crossings, such that the subdrawing of  $G$  is plane. An optimal solution to this problem approximates the crossing number of the graph  $G + F$ .

Finding an exact solution to MEI is NP-hard for general  $F$ , but linear time solvable for the special case of  $|F| = 1$  (SODA 01, Algorithmica) or when all of  $F$  are incident to a new vertex (SODA 09).

The complexity for general  $F$  but with constant  $k = |F|$  was open, but algorithms both with relative and absolute approximation guarantees have been presented (SODA 11, ICALP 11). We show that the problem is fixed parameter tractable (FPT) in  $k$  for biconnected  $G$ , or if the cut vertices of  $G$  have degrees bounded by a constant. We give the first exact algorithm for this problem; it requires only  $\mathcal{O}(|V(G)|)$  time for any constant  $k$ .

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# 1 Introduction

The crossing number  $\text{cr}(G)$  of a graph  $G$  is the minimum number of pairwise edge crossings in a drawing of  $G$  in the plane. Finding the crossing number of a graph is one of the most prominent combinatorial optimization problems in graph theory and is NP-hard already in very restricted cases, e.g., even when considering a planar graph with one added edge [5] (cf. the MEI problem for  $k = 1$  later). The problem has been vividly investigated for over 60 years, but there is still surprisingly little known about it; see e.g. [30] for an extensive reference. While in general, there exists a  $c > 1$  such that the crossing number cannot be approximated within a factor  $c$  in polynomial time [3], several approximation algorithms arose for special graph classes.

For general graphs with bounded degree, there is an algorithm that approximates the quantity  $n + \text{cr}(G)$  instead, giving an approximation ratio of  $\mathcal{O}(\log^2 n)$  [1, 16]. A sublinear approximation factor of  $\tilde{\mathcal{O}}(n^{0.9})$  for  $\text{cr}(G)$  in the bounded-degree setting was given in an involved algorithm [12]. We know constant factor approximations for bounded-degree graphs that are embeddable in some higher surface [17, 22, 24], or that have a small set of graph elements whose deletion leaves a planar graph—removing and re-inserting these elements can give strong approximation bounds such as [4, 11, 13, 23].

In this paper, we follow the latter idea and concentrate on the *Multiple Edge Insertion* problem  $\text{MEI}(G, F)$ , to be formally defined in the section hereafter. Intuitively, we are given a planar graph  $G$ , and ask for the best way (in terms of total crossing number) to planarly draw  $G$  and insert a set of new edges  $F$  into  $G$  such that the final drawing of  $G + F$  (i.e., of the graph including the new edges of  $F$ ) restricted to  $G$  remains planar.

This problem is polynomial-time solvable for  $|F| = 1$  [20] and in the case when all edges of  $F$  are incident to a common vertex [9], but NP-hard for general  $F$  [33]. Moreover, an exact or at least approximate MEI solution constitutes an approximation for the crossing number of the graph  $G + F$  [11]. Considering constant  $k := |F|$ , there have been two different approximation approaches [13] and [10]; the former one directly targets the crossing number and achieves only a relative approximation guarantee for MEI; the latter one first specifically attains an approximation of MEI with only an additive error term, and then uses [11] to deduce a crossing number approximation. While the former one is not directly practical, the latter algorithm [10] in fact turns out to be one of the best choices to obtain strong upper bounds in practice [8].

In this paper, we show that for every constant  $k$  and under mild connectivity assumptions, there is an exact linear time algorithm, which has so far been an open problem even for  $k = 2$ . In terms of parameterized complexity, our algorithm is in FPT with the parameter  $k = |F|$ .

**Theorem 1.** *Let  $G$  be a planar connected graph on  $n$  vertices, and  $F$  a set of  $k$  new edges (vertex pairs, in fact) where  $k$  is a constant. If  $G$  is biconnected, or the maximum degree of the cut vertices of  $G$  is bounded by a constant, then the problem  $\text{MEI}(G, F)$  is solvable in  $\mathcal{O}(n)$  time.*

We also mention that while the crossing number itself is in FPT w.r.t. the objective value [18, 26], already a planar graph with one added edge may have unbounded crossing number.

Both the aforementioned absolute MEI-approximation [10] and our new approach can use [11] to obtain the same relative ratio for approximating the crossing number. However, our new approach does so without any additional additive term:

**Corollary 2.** *Using the theorem relating an optimum  $\text{MEI}(G, F)$  solution to the crossing number of the graph  $G + F$  [11], Theorem 1 gives a polynomial time  $k\Delta$ -approximation for the crossing number of  $G + F$  with constant  $k = |F|$ , where  $G$  is a planar graph and  $\Delta$  is its maximum degree.*

**Organization.** After formally defining our setting in the next section, we will concentrate on the still NP-hard problem *Rigid MEI* in Section 3, i.e., MEI under the restriction that the planar embedding of  $G$  is fixed. In Section 4, this algorithm is at the core of a dynamic programming over a decomposition tree of  $G$ , in order to obtain an FPT algorithm for the general MEI, i.e., when any planar embedding of  $G$  is allowed. The latter constitutes the result for Theorem 1.

## 2 Preliminaries

We use the standard terminology of graph theory. By default, we use the term *graph* to refer to a loopless multigraph, i.e., we allow parallel edges but no self-loops. If there is no danger of confusion, we denote an edge with the ends  $u$  and  $v$  chiefly by  $uv$ .

A *drawing* of a graph  $G = (V, E)$  is a mapping of the vertices  $V$  to distinct points on a surface  $\Sigma$ , and of the edges  $E$  to simple (polygonal) curves on  $\Sigma$ , connecting their respective end points but not containing any other vertex point. Unless explicitly specified, we will always assume  $\Sigma$  to be the plane (or, equivalently, the sphere). A *crossing* is a common point of two distinct edge curves, other than their common end point. Then, a drawing is *plane* if there are no crossings. *Plane embeddings* form equivalence classes over plane drawings, in that they only define the cyclic order of the edges around their incident vertices (and, if desired, the choice of the outer, infinite face). A *planar* graph is one that allows a plane embedding. A *plane* graph is an embedded graph, i.e., a planar graph together with a planar embedding.

Given a drawing  $D$  of  $G$ , let  $\text{cr}(D)$  denote the number of pairwise edge crossings in  $D$ . The *crossing number* problem asks for a drawing  $D^\circ$  of a given graph  $G$  with the least possible number  $\text{cr}(D^\circ) =: \text{cr}(G)$ . By saying “pairwise edge crossings” we emphasize that we count a crossing point  $x$  separately for every pair of edges meeting in  $x$  (e.g.,  $\ell$  edges meeting in  $x$  give  $\binom{\ell}{2}$  crossings).

**Definition 3** (Multiple edge insertion, MEI and rigid MEI).

Consider a planar, connected graph  $G$  and a set of edges (vertex pairs, in fact)  $F$  not in  $E(G)$ . We denote by  $G + F$  the graph obtained by adding  $F$  to the edge set of  $G$ .

Let  $G_0$  be a planar embedding of  $G$ . The *rigid multiple edge insertion* problem  $\text{r-MEI}(G_0, F)$  is to find a drawing  $D$  of the graph  $G + F$  with minimal  $\text{cr}(D)$  such that the restriction of  $D$  to  $G$  is the plane embedding  $G_0$ . The attained number of crossings is denoted by  $\text{r-ins}(G_0, F)$ .

The *multiple edge insertion* problem  $\text{MEI}(G, F)$  is to find an embedding  $G_1$  of  $G$  (together with the subsequent drawing  $D$  as above), for which  $\text{r-MEI}(G_1, F)$  attains the minimum number of crossings. The latter is denoted by  $\text{ins}(G, F)$ .  $\diamond$

Herein, we will also deal with the *weighted* crossing number, i.e., we have edge weights  $w: E(G) \rightarrow \mathbb{N}_+ \cup \{\infty\}$ , and a crossing between two edges  $e_1, e_2$  accounts for the amount of  $w(e_1) \cdot w(e_2)$  in the above crossing functions. Specially, for the MEI problem variants, we shall consider integer weights on the edges of  $G$  but not on  $F$  (i.e., the weight on  $F$  is always 1). Although this is not a noteworthy strengthening of Theorem 1 by itself, the weights on  $E(G)$  will be useful in the recursive processing of the non-rigid case, cf. Section 4.

Given a plane embedding  $G_0$  of  $G$ , we define its (geometric) *dual*  $G_0^*$  as the embedded multigraph that has a (dual) vertex for each face in  $G_0$ ; dual vertices are joined by a (dual) edge for each (primal) edge shared by their respective (primal) faces. The *weight* of a primal edge gives rise to the *length* (of same value) of its dual edge. The cyclic order of the (dual) edges around any common incident (dual) vertex  $v^*$ , is induced by the cyclic order of the (primal) edges around the (primal) face corresponding to  $v^*$ .

We refer to a path/walk in  $G_0^*$  as to a *dual path/walk* in  $G_0$ , and we speak about a *dual path/walk*  $\pi$  in  $G_0$  *between vertices*  $u, v$  if the  $\pi$  starts in a face incident with  $u$  and ends in a face incident with  $v$ . We shortly say a *route from*  $u$  *to*  $v$  (a  $u$ - $v$  *route*) to mean a dual walk between vertices  $u, v$ .

For any drawing  $D$ , let  $\text{cr}_D(X, Y)$  denote the number of crossings between edges of  $X$  and edges of  $Y$  in  $D$ , and let  $\text{cr}_D(X) := \text{cr}_D(X, X)$ . It is well established that the search for an optimal solution to the crossing number problem can be restricted to so-called *good* drawings: any pair of edges crosses at most once, adjacent edges do not cross, and there is no point that is a crossing of three or more edges. A simple extension of this finding to the setting of MEI is presented next, in Lemma 4.

The following technical results will be used to restrict how “complicated” drawings of the edges of  $F$  may look in an optimal solution of a  $\text{r-MEI}(G, F)$  or  $\text{r-MEI}(G, F)$  instance. Note that, although both the claims are formulated for the rigid version, they easily imply the same for the ordinary (non-rigid) MEI problem.

**Lemma 4.** *Consider a (weighted) instance  $\text{r-MEI}(G, F)$  of the rigid MEI problem. In any optimal solution of  $\text{r-MEI}(G, F)$ , any two edges of  $F$  cross at most once, and they have no crossing if they share a common endvertex. Moreover, if the weights of the edges in  $F$  equal 1 and there exists a drawing  $D$  of  $G + F$  such that  $\text{cr}_D(E(G), F) = c$ , then  $\text{r-ins}(G, F) \leq c + \binom{|F|}{2}$ .*

*Proof.* The proof simply repeats, for this special case of rigid MEI, the folklore “arc exchange” argument from the crossing number theory. For the second claim, we observe that since the edge weights in  $F$  are all 1, it is  $\text{cr}_D(F, F) \leq \binom{|F|}{2}$ .  $\square$

**Corollary 5.** *Consider a (weighted) instance  $\text{r-MEI}(G, F)$  such that the weights of the edges in  $F$  equal 1, and let  $f \in F$ . Assume that  $D_1$  and  $D_2$  are two drawings of  $G + F$  such that  $D_1 - f$  is identical to  $D_2 - f$ , and that  $\text{cr}_{D_1}(E(G), \{f\}) - \text{cr}_{D_2}(E(G), \{f\}) > \binom{|F|}{2}$ . Then  $\text{cr}(D_1) > \text{r-ins}(G, F)$ , i.e.,  $D_1$  is not an optimal solution of  $\text{r-MEI}(G, F)$ .*

This claim might look rather weak at first sight, with respect to the required large difference  $d := \text{cr}_{D_1}(f, E(G)) - \text{cr}_{D_2}(f, E(G))$ . However, one can actually easily construct examples in which  $d = \Omega(|F|)$  and yet  $D_1$  is an optimal solution to  $\text{r-MEI}(G, F)$ .

*Proof.* Let  $E = E(G)$ ,  $k = |F|$  and  $F' = F \setminus \{f\}$ . Using Lemma 4, we estimate

$$\begin{aligned} \text{r-ins}(G, F) &\leq \text{cr}_{D_2}(E, F) + \binom{k}{2} = \text{cr}_{D_2}(E, F') + \text{cr}_{D_2}(E, \{f\}) + \binom{k}{2} \\ &= [\text{cr}_{D_1}(E, F') + \text{cr}_{D_1}(E, \{f\})] - \text{cr}_{D_1}(E, \{f\}) + \text{cr}_{D_2}(E, \{f\}) + \binom{k}{2} \\ &< \text{cr}_{D_1}(E, F) + 0 \leq \text{cr}(D_1). \end{aligned}$$

$\square$

### 3 Rigid MEI

In this section we give an FPT algorithm for solving the rigid version  $\text{r-MEI}(G, F)$ , parameterized by  $k = |F|$ .  $G$  is hence a plane graph (i.e., with a fixed embedding) throughout this section. Recall that the  $\text{r-MEI}(G, F)$  problem is NP-hard [33] for unrestricted  $k$ .

We first illustrate the simple cases. Solving  $\text{r-MEI}(G, \{uv\})$ , the fixed embedding edge insertion problem with  $k = 1$ , is trivial. Augment dual  $G^*$  with edges of length 0 between the terminals

$u, v$  (technically, new vertices in  $G^*$ ) and their respective incident faces (vertices in  $G^*$ ), to suit the above definition of a  $u$ - $v$  route in  $G$ . *Realizing* a route for  $uv$  means to draw  $uv$  along it within  $G$ . If the shortest route has length  $\ell$ , realizing it attains  $\text{r-ins}(G, \{uv\}) = \ell$ , the smallest number of crossings in the rigid MEI setting.

For  $k \geq 2$ , the situation starts to be more interesting: not every pair of shortest routes gives rise to an optimal solution of  $\text{r-MEI}(G, F)$  since there might arise a crossing between the two edges of  $F$ . The question, for  $k = 2$ , is whether some pair of shortest routes of the two edges in  $F$  can avoid crossing each other. Since it is generally not feasible to enumerate all shortest routes, we cannot check this by brute force and a more clever approach is needed. Even worse, for larger values of  $k$  we can encounter situations in which optimal solutions of  $\text{r-MEI}(G, F)$  draw edges of  $F$  quite far from their individual shortest routes (in order to avoid crossings with other edges of  $F$ ).

On a very high level, our approach to finding a drawing  $D$  of  $G + F$  that is an optimal solution to  $\text{r-MEI}(G, F)$ , can be described as follows:

- (I) We guess, for each pair  $f, f' \in F$ , whether  $f$  and  $f'$  will cross each other in  $D$ . Since  $k = |F|$  is a parameter, all the possibilities can be enumerated in FPT time.
- (II) Let  $X \subseteq \binom{F}{2}$  be a (guessed) set of pairs of edges of  $F$ . We find a collection of shortest routes for the edges of  $F$  in  $G$  under the restriction that exactly the pairs in  $X$  cross;  $D$  is obtained by inserting the edges of  $F$  along their computed routes. As we will see, we may restrict our attention only to routes pairwise crossing at most once.
- (III) We select  $D$  which minimizes the sum of  $|X|$  and of the lengths of the routes found above.

### 3.1 Handling path homotopy of routes

The core task of the scheme (I)–(III) is to find a collection of shortest routes under the restriction that every route avoids crossing certain other routes (note; none of these routes are fixed in advance). Although this problem may seem equivalent, in the dual, to the notoriously hard problem of shortest disjoint paths in planar graphs [14, 28], this is fortunately not the case since our routes may freely share their sections as long as they do not cross. We give a solution of the core task which is greedy in the sense that each route of  $F$  in  $G$  is minimized regardless of the other routes of  $F$ . The key to this solution is the concept of a *path homotopy* in the plane.

In a brief and rather informal topological view, consider the sphere with a finite set of point obstacles. Two simple curves  $\alpha, \alpha'$  with the same endpoints are *homotopic* if there exists a homeomorphism (a continuous deformation) of  $\alpha$  to  $\alpha'$  that fixes the endpoints and otherwise avoids all the obstacles. For example, if  $\alpha, \alpha'$  are disjoint except at the common ends, then they are homotopic if and only if one of the two open regions bounded by  $\alpha \cup \alpha'$  is obstacle-free. In our case, the *obstacles* are the ends  $V(F)$  of the edges of  $F$  (as given by the fixed embedding of  $G$ ), where each endpoint is “blown up” into a small open disc. Then, given the homotopy classes  $\text{hom}(\alpha), \text{hom}(\beta)$  of two curves  $\alpha, \beta$ , one can decide whether  $\alpha$  and  $\beta$  are “forced to cross”—although,  $\alpha$  and  $\beta$  may cross if they are not forced to, such an unforced crossing can as well be avoided in our case.

Instead of the above classical algebraic-topology setting of homotopies, in this paper we prefer to deal with path homotopy in a combinatorial setting. This setting is closely inspired by the discrete-geometry view of boundary-triangulated 2-manifolds by Hershberger and Snoeyink [21]. In the first step, we “triangulate” the point set  $V(F)$  (our obstacles) using transversing paths in the embedding  $G$ . A *transversing path* between vertices  $x, y$  of  $G$  is a path whose ends are  $x, y$  and whose internal vertices subdivide some edges of  $G$ . Let  $T$  be the union of these transversing paths and  $G'$  denote the corresponding subdivision of  $G$ . In order to avoid a terminology clash with graph triangulations, we will call  $T$  in the pair  $(G', T)$  a *trinet* of  $G$ . Formally (where  $V(F) = N$ ):

**Definition 6** (Trinet). Let  $G$  be a connected plane graph and  $N \subseteq V(G)$ ,  $|N| \geq 4$ . A plane graph  $T$  such that  $V(T) \cap V(G) = N$  is called a *trinet* of  $G$  if the following holds:

- a)  $T$  is a subdivision of a 3-connected plane triangulation on the vertex set  $N$  (in particular, every face of  $T$  is incident with precisely three vertices of  $N$ ), and
- b) there exists a subdivision  $G'$  of  $G$  such that  $V(G') \setminus V(G) = V(T) \setminus N$ ,  $E(G') \cap E(T) = \emptyset$  and the union  $G' \cup T$  is a plane embedding.

The pair  $(G', T)$  is a *full trinet* of  $G$ . The vertices in  $N(T) := N$  are called *trinodes* of  $T$ , the maximal paths in  $T$  internally disjoint from  $N$  are *triedges* and their set is denoted by  $I(T)$ , and the faces of  $T$  are *tricells*. Note that the triedges of  $T$  are transversing paths of  $G$ . We refer to Figure 1 for a brief illustration of this definition.  $\diamond$

Second, we focus on terms related to path homotopy in a full trinet  $(G', T)$  of a plane graph  $G$ . Moreover, while we have implicitly perceived a route of  $uv$  in  $G$  (i.e., a dual walk from  $u$  to  $v$ ) as an arc drawn from  $u$  to  $v$ , we would also like to describe a topological “alley” for all  $u$ – $v$  arcs of a similar kind (and same number of crossings) in the embedding  $G' \cup T$ . With it we gain combinatorial abstraction and will later be able to avoid unforced crossings with other routes.

**Definition 7** (Alley and  $T$ -sequence). Let  $(G', T)$  be a full trinet of a plane graph  $G$ . Consider a route  $\pi$  between  $u, v \in V(G)$  in the graph  $G' \cup T$ . Then  $V(\pi) = \{\phi_0, \phi_1, \dots, \phi_m\}$  where each dual vertex  $\phi_i$  of  $\pi$  is an open face of  $G' \cup T$ . Let these faces  $(\phi_0, \phi_1, \dots, \phi_m)$  be ordered along  $\pi$  such that  $\phi_0$  is incident to  $u$  and  $\phi_m$  incident to  $v$ . Let  $(e_1, e_2, \dots, e_m) \subseteq E(G' \cup T)$  be the sequence of the primal edges of the dual edges of  $\pi$ , ordered from  $\phi_0$  to  $\phi_m$ . As a point set, each edge  $e_i$  is considered without the endpoints.

- a) The union  $\{u, v\} \cup \bigcup_{i=0}^m \phi_i \cup \bigcup_{i=1}^m e_i$  is called the *alley* of  $\pi$  (or, an *alley between  $u, v$* ).
- b) Let  $(e'_1, \dots, e'_\ell) \subseteq (e_1, e_2, \dots, e_m)$  be the restriction to  $E(T)$ , and let  $(p_1, p_2, \dots, p_\ell) \subseteq I(T)$  be the sequence of triedges such that  $p_i$  contains the edge  $e'_i$  for  $i = 1, \dots, \ell$ . Then  $(p_1, p_2, \dots, p_\ell)$  is called the  *$T$ -sequence of  $\pi$*  from  $u$  to  $v$  (or, of the corresponding alley from  $u$  to  $v$ ).  $\diamond$

A route  $\pi$  *crosses* a triedge  $p$  if the alley of  $\pi$  contains one of the  $G'$ -edges forming  $p$ . The  $T$ -sequence of  $\pi$  hence describes the unique order (with repetition) in which its alley crosses the triedges of  $T$ . Usually, we shall consider only the case of  $u, v \in N(T)$ .

A route may, in general, cross the same triedge  $q$  many times in one place (switching “there and back”). However, such a situation may be easily smoothed down to one or no crossing, and this can be formalized by the notion of reducing a  $T$ -sequence as follows: if  $S = (p_1, p_2, \dots, p_\ell)$  is a  $T$ -sequence such that  $p_i = p_{i+1}$  for some  $1 \leq i < \ell$ , then the subsequence  $S' = (p_1, \dots, p_{i-1}, p_{i+2}, \dots, p_\ell)$  is called a one-step reduction of  $S$ . A subsequence  $S^* \subseteq S$  is a *reduction* of  $S$  (or  $S$  *reduces to  $S^*$* ) if  $S^*$  results from a sequence of one-step reductions of  $S$ .

It comes as no surprise that  $T$ -sequences are closely related to the homotopy concept:

*Remark 8.* Consider a trinet  $T$  in the sphere. One can show that two arcs with the same fixed endpoints are path-homotopic (in the sphere with the obstacles formed by the trinodes of  $T$ ) if, and only if, their  $T$ -sequences can be reduced to the same subsequence. However, since we are not going to directly use this fact, we refrain from giving this as a formal statement in the short paper.

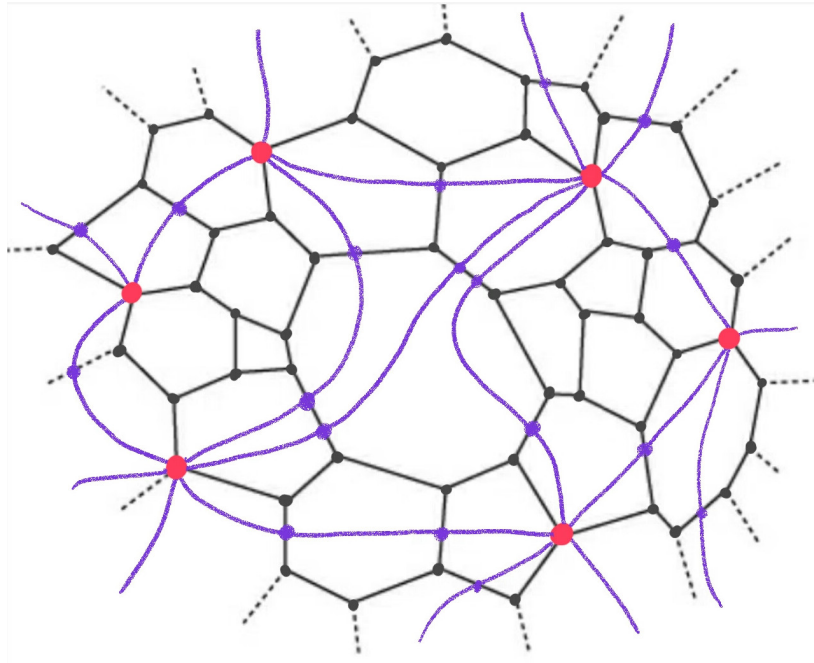


Figure 1: An example of a trinet in a plane graph (see Definition 6): the underlying plane graph  $G$  is in black, the trinodes in thick red and the triedges in blue.

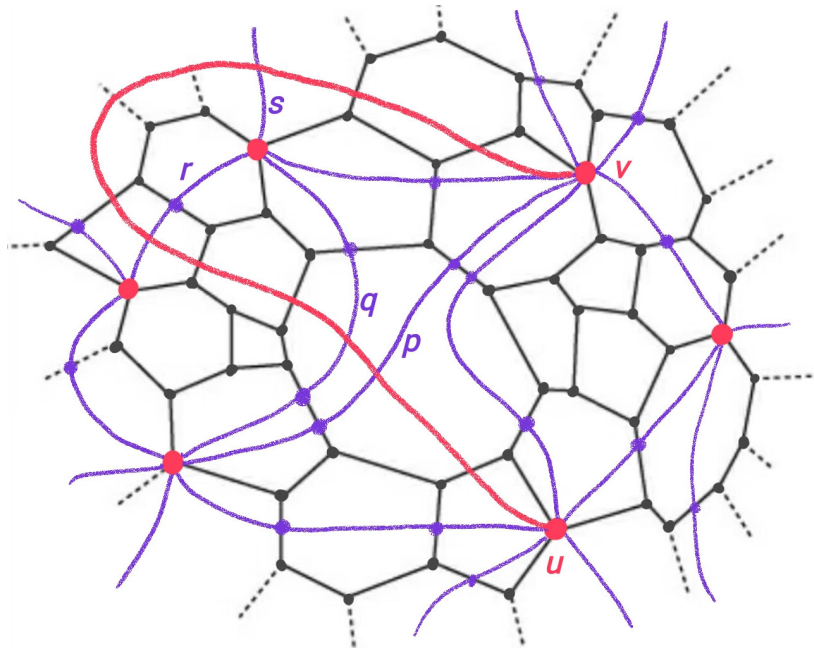


Figure 2: An example (see Definition 7): the  $T$ -sequence of the  $u$ - $v$  route depicted in red is  $(p, q, r, s)$  from  $u$  to  $v$ . It is a proper  $T$ -sequence from  $u$  to  $v$  (Definition 10).

### 3.2 $T$ -sequences of potential shortest routes

Our goal is to look for shortest routes in  $G$  of a given homotopy, and we slightly generalize the setting to allow for a connected plane graph  $G$  with *edge weights*  $w: E(G) \rightarrow \mathbb{N}_+ \cup \{\infty\}$ . For a full trinet  $(G', T)$  of  $G$ , we define the edge weights of  $G' \cup T$  as follows:  $w'(p) := 0$  for all  $p \in E(T)$  and  $w'(e') := w(e)$  where  $e' \in E(G')$  is obtained by subdividing  $e \in E(G)$ . This  $w'$  is the *weight induced by  $w$*  in the trinet  $(G', T)$ . We give the same weights  $w'$  also to the edges of the geometric dual of  $G' \cup T$ . If  $\alpha$  is the alley of a route  $\pi$  between vertices  $x, y$  in  $G' \cup T$ , then the *length of  $\alpha$*  equals the length of  $\pi$ , i.e., the sum of the  $w'$ -weights of the dual edges of  $\pi$ .

With the help of the framework developed in the previous section, we can now give an (again informal) high-level refinement of our solution steps (I)–(III) of  $\text{r-MEI}(G, F)$  as follows:

- (IV) Consider a trinet  $T$  of  $G$  on the trinodes  $V(F)$ . If we fix a (realizable)  $T$ -sequence  $S$ , then we can use established tools, namely an adaptation of the idea of the funnel algorithm [6, 29], to efficiently compute a shortest alley among those having the same  $T$ -sequence  $S$ . For  $uv \in F$  of weight 1, if we compute an alley  $\alpha$  between  $u, v$  of length  $\ell$ , then we can easily draw the new edge  $uv$  as an arc in  $\alpha$  with  $\ell$  weighted crossings.
- (V) Suppose that, for  $i = 1, 2$ ,  $\alpha_i$  is a shortest alley between  $x_i$  and  $y_i$  having the  $T$ -sequence  $S_i$ . Then, as detailed later in Lemma 17 and also Claim 20, we can decide from *only*  $S_1, S_2$  whether there exist arcs from  $x_1$  to  $y_1$  in  $\alpha_1$  and from  $x_2$  to  $y_2$  in  $\alpha_2$ , which do not cross (note that  $\alpha_1 \cap \alpha_2$  may be nonempty and yet there may exist such a pair of non-crossing arcs). Moreover, if the two arcs cross then it should be only once.
- (VI) Consequently, it will be enough to loop through all “suitable”  $T$ -sequences for every edge of  $F$  and independently perform the steps (IV), (V) for each combination of them, in order to get an optimal solution of  $\text{r-MEI}(G, F)$  as in (III). The point is to bound the number of  $T$ -sequences that have to be considered, in terms of only the parameter  $k = |F|$ .

We first resolve the last point (VI) which is a purely mathematical question. In order to achieve the goal, we will build a special trinet of  $G$  along (at least locally) shortest dual paths between the trinodes in  $G$  (Definition 9). Then we will be able to restrict our attention to special  $T$ -sequences of bounded length (Definition 10 and Lemma 12).

**Definition 9** (Shortest-spanning trinet). Let  $(G', T)$  be a full trinet of a plane graph  $G$ , and let the weights  $w'$  in  $(G', T)$  be induced by weights  $w$  in  $G$ . For a triedge  $q \in I(T)$ , every internal vertex  $t$  of  $q$  is incident with two edges  $e, e'$  of  $G'$  of weight  $w'(e) = w'(e')$  which we call the weight of  $t$ . The *transversing weight of  $q$*  equals the sum of the weights of the internal vertices of  $q$ .

A triedge  $q \in I(T)$  between trinodes  $x, y$  is *locally-shortest* if the transversing weight of  $q$  is equal to the length of a shortest dual path  $\pi$  in  $G' \cup T$  between  $x, y$ , such that  $\pi$  is contained in(!) the union of the two tricells incident with  $q$ . Similarly,  $q$  is *globally-shortest* if the transversing weight of  $q$  is equal to the dual distance between  $x, y$  in  $G' \cup T$ .

We say that  $T$  has the *shortest-spanning property* if every triedge in  $I(T)$  is locally-shortest, and there exists a subset of triedges  $J \subseteq I(T)$  forming a connected subgraph of  $T$  spanning all the trinodes such that every triedge in  $J$  is globally-shortest.  $\diamond$

**Definition 10** (Proper  $T$ -sequence). Consider a trinet  $T$  and trinodes  $u \neq v \in N(T)$ . A *nonempty* sequence  $S = (p_1, p_2, \dots, p_m) \subseteq I(T)$  of triedges of  $T$  (repetition allowed) is a *proper  $T$ -sequence from  $u$  to  $v$*  if the following holds:  $u$  is disjoint from  $p_1$  but there exists a tricell  $\theta_0$  incident with both  $u$  and  $p_1$ ,  $v$  is disjoint from  $p_m$  but there exists a tricell  $\theta_m$  incident with both  $v$  and  $p_m$ , and each two consecutive triedges  $p_i, p_{i+1}$  are distinct and incident to a common tricell  $\theta_i$  for  $1 \leq i < m$ . *Empty  $S$*  is a *proper  $T$ -sequence from  $u$  to  $v$*  if  $u, v$  are incident to a common tricell  $\theta_0$ .  $\diamond$



Recalling that  $T$  is a subdivision of a triangulated graph, we immediately get the following:

**Claim 11.** *For every proper nonempty  $T$ -sequence  $S$ , the sequence of tricells  $(\theta_0, \theta_1, \dots, \theta_m)$  as in Definition 10 is uniquely determined by  $T$  and  $S$ .  $\square$*

For empty proper  $S$ , a tricell  $\theta_0$  from Definition 10 is not unique (there are two choices for it). However, since any of the two choices of  $\theta_0$  incident with both  $u, v$  will work for us in the same way in a shortest-spanning trinet, we simply make an arbitrary deterministic choice of  $\theta_0$  for  $S = \emptyset$  and extend the scope of Claim 11 also to empty proper  $T$ -sequences.

What follows is the crucial finding that makes our algorithm to work:

**Lemma 12.** *Consider an instance  $\text{r-MEI}(G, F)$  where  $G$  is a connected plane graph. Let  $(G', T)$  be a full trinet of  $G$  having the shortest-spanning property. There exists a set  $\{\pi_f : f \in F\}$  where  $\pi_f$  for  $f = uv$  is a route in  $G' \cup T$  between the trinodes  $u, v$ , such that the following hold:*

- a) *There exists an optimal drawing  $D$  of  $G + F$  with  $\text{r-ins}(G, F)$  crossings such that each edge  $f \in F$  is drawn in the alley of  $\pi_f$ , and no two edges of  $F$  cross each other more than once.*
- b) *The  $T$ -sequence  $S_f$  of each  $\pi_f$  is a proper  $T$ -sequence, and no triedge occurs in  $S_f$  more than  $8k^4$  times where  $k = |F|$ .*

Note that optimality of the number  $\text{r-ins}(G, F)$  ensures that, in a), no two edges of  $F$  cross in  $D$  unless they are forced to (by their given alleys).

*Proof.* In the scope of this proof, we shall use the following special terminology and notation. For simplicity, we use the symbol  $f$  both for an edge  $f \in F$  and for the arc representing  $f$  in a specific drawing of  $G + F$  (more generally, of  $(G' + F) \cup T$ ). We similarly consider a triedge  $p \in I(T)$  also as the arc representing  $p$  in  $G'$ . If  $x, y$  are two points on any arc  $b$ , then let  $b[x, y]$  denote the section of the arc from  $x$  to  $y$ .

Consider the given shortest-spanning trinet and the corresponding plane embedded graph  $G' \cup T$  with edge weights  $w'$  induced by given  $w$  of  $G$ . We will implicitly assume that every arc  $a$  drawn in  $G' \cup T$  avoids crossing the vertices and  $a$  intersects  $G' \cup T$  in finitely many points, i.e., the embedding and the arcs may be restricted to polygonal lines. For any arc  $b$  with the ends  $u, v$ , we define the  $T$ -sequence of  $b$  from  $u$  to  $v$  as the sequence (with repetition) in which  $b$  intersects the triedges of  $T$ . We define the *transversing weight* of  $b$ , shortly *t-weight*, as the sum of the  $w'$ -weights of the edges of  $G' \cup T$  crossed by  $b$ , and denote it by  $t_{w'}(b)$ .

We choose an optimal drawing  $D$  of  $G' + F$  which, at the same time, minimizes the combined length of the  $T$ -sequences of the edges of  $F$ , i.e., the number of crossings between  $F$  and the trinet  $T$ . Recall from Lemma 4: any two edges of  $F$  cross at most once, and they have no crossing if they share a common endvertex. For  $f \in F$ , let  $S_f = (p_f^1, \dots, p_f^{m_f})$  be the  $T$ -sequence of  $f$  and let  $x_f^1, \dots, x_f^{m_f}$ , respectively, denote the points at which the arc of  $f$  intersects the triedges of  $T$ . The first task is to prove that each  $S_f$  is a proper  $T$ -sequence.

We start with a stronger technical claim: if, for some  $f \in F$  and  $j > i$ , it is  $p_f^i = p_f^j$  and the simple loop  $a := p_f^i[x_f^i, x_f^j] \cup f[x_f^i, x_f^j]$  is contractible (i.e., with no trinode inside), then we get a contradiction to the choice of  $D$  above. Indeed, we may assume that  $f$  and  $j > i$  are chosen such that  $a$  encloses minimal area in the drawing  $D$ . By the minimality of  $a$ , no triedge crosses the interior of  $f[x_f^i, x_f^j]$  twice (all  $p_f^{i+1}, \dots, p_f^{j-1}$  are distinct). However, since the interior enclosed by  $a$  contains no trinode, the previous implies that no triedge other than  $p_f^i$  may intersect  $a$ , and so  $j = i + 1$ . Consequently, since the triedge  $p_f^i$  is locally shortest in  $T$ , the t-weights of the considered section satisfy  $t_{w'}(p_f^i[x_f^i, x_f^j]) \leq t_{w'}(f[x_f^i, x_f^j])$ . If we re-route  $f$  closely along  $p_f^i[x_f^i, x_f^j]$  (without

crossing  $p_f^i$ ), then this change does not increase the crossing number by the inequality of t-weights, but the  $T$ -sequence of  $f$  gets shorter (see in Figure 3). Hence, it contradicts our choice of  $D$ .

Now we get back to  $S_f$  being a proper  $T$ -sequence. If  $S_f$  is empty, then the statement is trivial. If  $S_f$  contains consecutive repeated triedge  $p_f^i = p_f^{i+1}$  for some  $1 \leq i < m_f$ , then the above contradiction directly applies. Assume now that  $f = uv$  and the triedge  $p_f^1$  is incident with the starting trinode  $u$ . Then we can apply the same contradiction to the contractible loop  $a := p_f^1[u, x_f^1] \cup f[u, x_f^1]$ . The remaining properties of proper  $T$ -sequences follow trivially.

The last and most difficult step is to prove that no triedge repeats in  $S_f$  too many times, for each  $f \in F$ . Again, if  $p_f^i = p_f^j = p'$  for  $i \neq j$ , then the simple loop  $a' := p'[x_f^i, x_f^j] \cup f[x_f^i, x_f^j]$  must be non-contractible, and so separating some pair of trinodes of  $T$  from each other. In particular, by assumed connectivity of  $G'$ , this implies that always  $t_{w'}(a') \geq 1$ . Since at most  $2k - 1$  globally-shortest tries of  $T$  span all the trinodes by Definition 9,  $a'$  (and consequently  $f[x_f^i, x_f^j]$ ) must cross at least one of them. Therefore, if a triedge  $p'$  repeats in  $S_f$  at least  $8k^4$  times, then there is a globally-shortest triedge  $p$  of  $T$  such that  $p$  repeats in  $S_f$  at least  $\ell \geq 8k^4/(2k - 1) > 4k^3$  times.

Let  $Y = (y_1, \dots, y_\ell)$  (a subsequence of  $(x_f^1, \dots, x_f^{m_f})$ ) be the ordered sequence of points in which the arc of  $f$  intersects the arc of the triedge  $p$ . We say that an index  $i \in \{2, \dots, \ell - 1\}$  is a *switchback* of  $Y$  if  $y_{i-1}, y_{i+1}$  both lie on the same side of  $y_i$  on  $p$ . Up to symmetry, let the points on  $p$  be ordered such that  $y_{i+1}$  lies between  $y_{i-1}, y_i$ . Since  $p$  is globally-shortest in  $T$ , we get (now regardless of contractibility of the induced loops)

$$t_{w'}(p[y_{i-1}, y_i]) \leq t_{w'}(f[y_{i-1}, y_i]),$$

and then

$$\begin{aligned} t_{w'}(f[y_{i-1}, y_{i+1}]) &\geq t_{w'}(p[y_{i-1}, y_i]) + t_{w'}(f[y_i, y_{i+1}]) \\ &= t_{w'}(p[y_{i-1}, y_{i+1}]) + t_{w'}(p[y_{i+1}, y_i]) + t_{w'}(f[y_i, y_{i+1}]) \\ &\geq t_{w'}(p[y_{i-1}, y_{i+1}]) + 1. \end{aligned}$$

Hence, if we locally re-route  $f$  along  $p[y_{i-1}, y_{i+1}]$ , then we save the amount of at least 1 in the crossings of  $f$  with  $E(G)$ . Note that this is not a contradiction to our choice of optimal drawing  $D$  yet since the change may introduce many new crossings of  $f$  with the rest of  $F$ . However, we cannot have more than  $\binom{k}{2}$  switchbacks in  $Y$  or we get a contradiction using Corollary 5. (Observe that this usage of the corollary here is the only reason why we are restricting ourselves to unweighted sets  $F$ .)

Since  $\ell \geq 4k^3$ , there is a consecutive subsequence  $Y' \subseteq Y$  of length  $\ell' \geq \ell/\binom{k}{2} - 1 > 8k$  without switchbacks. Without loss of generality, we assume  $Y' = (y_1, \dots, y_{\ell'})$ . Let  $g_i := f[y_i, y_{i+1}]$  and  $g_i^\circ := g_i \cup p[y_i, y_{i+1}]$ , for  $i \in \{1, \dots, \ell' - 1\}$ . As argued before, each  $g_i^\circ$  is a simple loop separating some pair of trinodes of  $T$ . Since no two edges of  $F$  cross more than once, there are at most  $k - 1$  indices  $i \in \{1, \dots, \ell' - 1\}$  such that  $g_i$  is crossed by another edge(s) of  $F$ .

Let  $x, y$  be the ends of the triedge  $p$ . Assume that we have  $i \neq j \in \{1, \dots, \ell' - 1\}$  such that neither of  $g_i^\circ, g_j^\circ$  separates  $x$  from  $y$ . Let  $Z_i \neq \emptyset$  denote the set of trinodes of  $T$  that are separated by  $g_i^\circ$  from  $x, y$ , and let  $Z_j$  be defined analogously. We claim that  $X_i \cap X_j = \emptyset$ . If not, then—up to symmetry— $g_j^\circ$  is separated from  $x, y$  by  $g_i^\circ$ , except a possibly shared section of  $p[y_i, y_{i+1}]$ . The former is impossible by the Jordan curve theorem and the latter would mean that there is a switchback between  $i$  and  $j$ , which is again a contradiction. Since there are at most  $2k - 2$  pairwise disjoint nonempty possibilities (e.g., singleton trinodes other than  $x, y$ ) for the sets  $X_i, X_j$ , at most  $2k - 2$  indices  $i \in \{1, \dots, \ell' - 1\}$  are such that  $g_i^\circ$  does not separate  $x$  from  $y$ .

Since  $\ell' \geq 8k$ , there exists a set of indices  $J \subseteq \{1, \dots, \ell' - 2\}$ ,  $|J| \geq \ell' - 2(k - 1 + 2k - 2) - 2 > 2k$ , such that for every  $j \in J$  both the arcs  $g_j, g_{j+1}$  are not crossed by other edges of  $F$  and both  $g_j^\circ, g_{j+1}^\circ$  separate  $x$  from  $y$ . Let  $f_0 := f[y_1, y_{\ell'}]$  and  $p_0 := p[y_1, y_{\ell'}]$ ; we get  $Y' \subseteq p_0$  since there are no switchbacks in  $Y'$ . Observe also that  $g_j^\circ \cap g_{j+1}^\circ = \{y_{j+1}\}$  since  $f$  is not self-intersecting and there is no switchback in  $Y'$ . Hence, up to symmetry,  $g_j^\circ$  separates  $x$  from  $g_{j+1}^\circ$ , and  $g_{j+1}^\circ$  separates  $g_j^\circ$  from  $y$ . It easily follows that  $g_j^\circ \cup g_{j+1}^\circ$  forms the boundary of an arc-connected region of  $\mathbb{R}^2 \setminus (f_0 \cup p_0)$  (a *face* of  $f_0 \cup p_0$ ). Since at most  $2k - 2$  of the faces of  $f_0 \cup p_0$  may contain a trinode of  $T$  other than  $x, y$ , there exists  $j \in J$  such that, in addition to the above properties of  $J$ , the face  $\sigma$  bounded by  $g_j^\circ \cup g_{j+1}^\circ$  contains no trinode (see in Figure 4).

Our goal now is to re-route  $f$  along  $p[y_j, y_{j+1}]$  (i.e., “replacing” the part  $g_j \subset f$ ). Again, since  $p$  is globally-shortest in  $T$ , this move does not increase the number of crossings of  $f$  with  $E(G)$ , and the  $T$ -sequence of  $f$  gets shorter. It remains to argue that we can avoid new crossings of  $f$  with  $F \setminus \{f\}$ . If any  $f' \in F$  crosses  $p[y_j, y_{j+1}]$  then, since  $\sigma$  contains no trinode,  $f'$  has to leave  $\sigma$  as well, and the only possibility is across  $p[y_{j+1}, y_{j+2}]$  by the previous assumptions. Consequently, such  $f'$  can be re-routed along  $p[y_j, y_{j+2}]$ , similarly to  $f$ , and no crossing with  $f$  is required (see again in Figure 4). Note, moreover, that even if two such edges  $f', f'' \in F$  cross each other in  $\sigma$ , there is no problem and they will cross in their new routing in the same way. We have again reached a contradiction to our choice of  $D$ .  $\square$

### 3.3 Shortest routes in a sleeve, and crossing of routes

Now, consider step (IV) of our outline—as mentioned before, we solve this step separately for each  $f = uv \in F$ . To recapitulate, for trinodes  $u, v$  of a trinet  $T$  of  $G$  and a given proper  $T$ -sequence  $S$  from  $u$  to  $v$ , the task is to find a shortest route from  $u$  to  $v$  among those having the same  $T$ -sequence  $S$ . We cannot, in general, completely avoid repeating triedges in  $S$  and tricells in the sequence  $(\theta_0, \theta_1, \dots, \theta_m)$  in Definition 10. To prevent related technical difficulties, we use a similar workaround as in [21]; “lifting” the respective sequence of tricells into a universal cover as follows.

**Definition 13** (Sleeve of a  $T$ -sequence). Let  $(G', T)$  be a full trinet of a plane graph  $G$ , and consider a proper  $T$ -sequence  $S = (p_1, p_2, \dots, p_m)$  from  $u$  to  $v$  determining the sequence of tricells  $(\theta_0, \theta_1, \dots, \theta_m)$  by Claim 11. For  $i = 0, 1, \dots, m$ , let  $L_i$  be a disjoint copy of the embedded subgraph of  $G' \cup T$  induced by  $\theta_i$ . Construct a plane graph  $L$  from the union  $L_0 \cup \dots \cup L_m$  by identifying, for  $j = 1, \dots, m$ , the copy of the triedge  $p_j$  in  $L_{j-1}$  with the copy of  $p_j$  in  $L_j$ . We call  $L$  the *sleeve* of  $S$  in the trinet  $(G', T)$ , and we identify  $u$  and  $v$  with their copies in  $L_0$  and  $L_m$ , respectively. We make the unique face of  $L$  that is not covered by a copy of any tricell of  $T$  the *outer face* of  $L$ .  $\diamond$

Observe that every route from  $u$  to  $v$  in  $G' \cup T$  having its  $T$ -sequence equal to  $S$  can be easily lifted into a corresponding  $u$ – $v$  route in the sleeve  $L$  of  $S$ . Conversely, any  $u$ – $v$  route in  $L$  avoiding the outer face and crossing the copies of triedges in  $L$  at most once each, can be obviously projected down to  $G' \cup T$  to make a route with the  $T$ -sequence equal to  $S$ . In fact, we prove that some shortest  $u$ – $v$  route in  $L$  must be of the latter kind, under the shortest-spanning property (cf. Definition 9).

**Lemma 14.** *Let  $(G', T)$  be a shortest-spanning full trinet of an edge-weighted plane graph  $G$ ,  $S$  a proper  $T$ -sequence between trinodes  $u, v$  of  $T$ , and let  $L$  be the sleeve of  $S$ . Let  $\ell$  be the length of a shortest route from  $u$  to  $v$  among those having the  $T$ -sequence  $S$ . Then,  $\ell$  is equal to the dual distance from  $u$  to  $v$  in  $L$  without the outer face. Furthermore, at least one of the  $u$ – $v$  routes of length  $\ell$  in  $L$  crosses the copy of each triedge from  $S$  in  $L$  exactly once.*

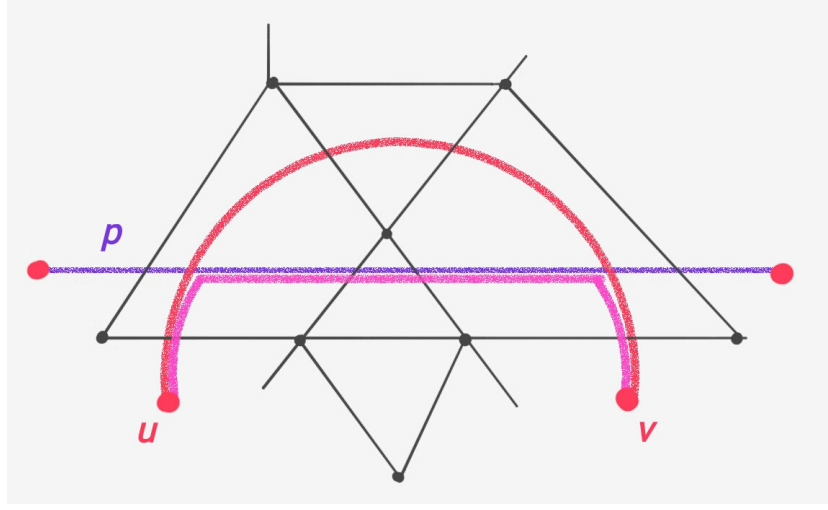


Figure 3: Two consecutive crossings of the arc of  $f = uv$  with a triedge  $p$  (which is locally-shortest) determine a contractible loop, and so  $f$  can be re-routed partly along  $p$  without inducing more crossings with  $G$  or with other edges of  $F$ .

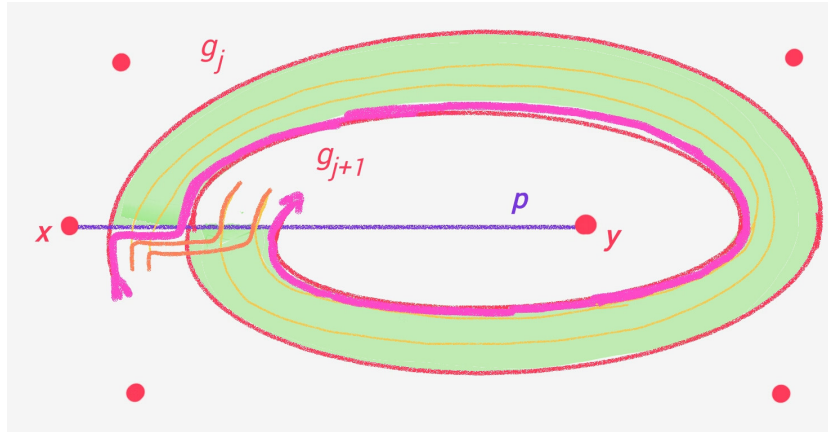


Figure 4: The face bounded by  $g_j^\circ \cup g_{j+1}^\circ$  ( $\subseteq f \cup p$ ) depicted in green: since there is no trinode in this face and neither  $g_j, g_{j+1}$  are crossed by other edges of  $F$ , it is possible to re-route  $f$  partly along  $p$  such that it avoids  $g_j$ . Other possible  $F$ -edges entering the green face through a section of  $p$  must leave at the other end, and hence can be re-routed similarly to  $f$ .

*Proof.* Let  $\pi$  be any shortest route from  $u$  to  $v$  in  $G' \cup T$  with the given  $T$ -sequence  $S$ . The copies of the faces and dual edges of  $\pi$  lifted into the sleeve  $L$  give a route  $\pi_L$  from  $u$  to  $v$  which avoids the outer face of  $L$ . Obviously, the length of  $\pi_L$  equals the length of  $\pi$ .

Conversely, we aim to show that some shortest  $u$ - $v$  route crosses the copy of each triedge of  $S$  in  $L$  exactly once. Assume a shortest route  $\pi_1$  of length  $\ell$  from  $u$  to  $v$  in  $L$  without the outer face. Recall from Definition 13 that  $L = L_0 \cup \dots \cup L_m$ . For  $S = (p_1, p_2, \dots, p_m)$ , let  $(p'_1, p'_2, \dots, p'_m)$  be the sequence of corresponding copies of the triedges of  $S$  in  $L$ , and let  $p'_0 := u, p'_{m+1} := v$ . Note that each  $p'_i, i \in \{1, \dots, m\}$ , connects two vertices of the outer face of  $L$ , and so  $p'_i$  separates  $p'_{i-1}$  from  $p'_{i+1}$ . In particular, every  $u$ - $v$  route in  $L$  which avoids the outer face must cross each of  $p'_1, \dots, p'_m$ .

Let  $i$  be the maximum index such that  $p'_i$  is crossed by  $\pi_1$  more than once. Then there is a subpath  $\sigma_1 \subseteq \pi_1$  stretching between two consecutive crossings of  $\pi_1$  with  $p'_i$  and contained in  $L_i$ . We turn  $\pi_1$  into  $\pi_2$  by re-routing the subpath  $\sigma_1$  along the boundary  $p'_i$  in  $L_{i-1}$ . Since  $p_i$  is locally-shortest in the trinet  $T$ , the length of  $\pi_2$  equals the length of  $\pi_1$ . By induction on the number of excess crossings of  $\pi_2$  with copies of the triedges, we can then get a  $u$ - $v$  route  $\pi_0$  of length  $\ell$  such that  $\pi_0$  crosses the copy of each triedge from  $S$  in  $L$  exactly once.

Finally, the route  $\pi_0$  projects down to a route of length  $\ell$  from  $u$  to  $v$  in  $G' \cup T$  having the  $T$ -sequence  $S$ .  $\square$

Regarding the shortest path computation, we note that if the edge-weights are given in unary or are bounded by a constant, a simple adaption of BFS achieves the job. Otherwise, we can use the algorithm of Klein et al. [27] since  $L$  is planar, or Thorup's algorithms [31] since we have integral weights. Altogether we obtain:

**Corollary 15.** *Let  $(G', T)$  be a shortest-spanning full trinet of an edge-weighted plane graph  $G$ , and  $S$  a proper  $T$ -sequence between trinodes  $u, v$  of  $T$ . A shortest  $u$ - $v$  route among those having the  $T$ -sequence  $S$  can be found in the geometric dual graph of the sleeve  $L$  of  $S$  in  $\mathcal{O}(|S| \cdot |N(T)| \cdot |V(G)|)$  time, using a linear time shortest path algorithm.*

Observe that in our case the term  $|S| \cdot |N(T)|$  is bounded by a function of  $k = |F|$ .

*Proof.* By Definition 6, the size of the full trinet  $(G', T)$  is  $\mathcal{O}(|N(T)| \cdot |V(G)|)$ , and the sleeve is composed of  $|S|$  copies of subgraphs of  $G' \cup T$ , and so  $\mathcal{O}(|S| \cdot |N(T)| \cdot |V(G)|)$  bounds the size of the sleeve  $L$ . Therefore, any linear time shortest path algorithm applicable in our situation, e.g. the aforementioned Klein et al. [27] or Thorup [31], does the job.  $\square$

Finally, it remains to address step (V). Consider a 4-tuple of distinct trinodes  $u, v, u', v'$ . Let  $\pi$  be a  $u$ - $v$  route and  $\pi'$  be a  $u'$ - $v'$  route. We say that an *arc*  $b$  follows the route  $\pi$  if  $b$  is contained in the alley of  $\pi$  and  $b$  intersects the faces forming the alley exactly in the order given by  $\pi$  (recall that a route is technically a dual walk and hence, possibly, some face might repeat in  $\pi$ ). We say that the pair of routes  $\pi, \pi'$  is *non-crossing*, if there exist a  $u$ - $v$  arc  $b$  following  $\pi$  and a  $u'$ - $v'$  arc  $b'$  following  $\pi'$  such that  $b \cap b' = \emptyset$ . In order to characterize possible non-crossing pairs of routes in terms of their  $T$ -sequences, we bring the following definition:

**Definition 16** (Crossing certificate). Let  $(G', T)$  be a full trinet of a plane graph  $G$ , and let  $\pi$  be a route from  $u$  to  $v$  and  $\pi'$  be a route from  $u'$  to  $v'$  in  $G' \cup T$ , where  $u, v, u', v'$  are distinct trinodes of  $T$ . Assume the  $T$ -sequences  $S = (p_1, \dots, p_n)$  of  $\pi$  and  $S' = (p'_1, \dots, p'_\ell)$  of  $\pi'$  are proper and let  $(\theta_0, \dots, \theta_n)$  and  $(\theta'_0, \dots, \theta'_\ell)$  be their tricell sequences by Claim 11. For technical reasons, let  $p_0 := u, p_{n+1} := v$  and  $p'_0 := u', p'_{\ell+1} := v'$ .

A *crossing certificate* for  $S, S'$  is a triple of indices  $(c, d, m)$  where  $c, d, m \geq 0, c + m \leq n, d + m \leq \ell$ , such that the following holds:

- a)  $\theta_{c+j} = \theta'_{d+j}$  for  $0 \leq j \leq m$ , but  $p_c \neq p'_d$  and  $p_{c+m+1} \neq p'_{d+m+1}$ ,
- b) the triple  $p_c, p'_d, p_{c+1}$  occurs around the tricell  $\theta_c = \theta'_d$  in the same cyclic orientation as the triple  $p_{c+m+1}, p'_{d+m+1}, p_{c+m}$  occurs around  $\theta_{c+m} = \theta'_{d+m}$ .

Furthermore, a crossing certificate for the same sequence  $S$  and the reversal of  $S'$  from  $v'$  to  $u'$  is also called a crossing certificate for  $S, S'$ .  $\diamond$

Definition 16 deserves a closer explanation. Assume that a crossing certificate satisfies  $0 < c < n$  and  $0 < d < \ell$ . Then all four elements  $p_c, p'_d, p_{c+1}, p'_{d+1}$  are triedges of the same tricell  $\theta_c = \theta'_d$ , and since  $p_{c+1} \neq p_c \neq p'_d \neq p'_{d+1}$ , we get  $p_{c+1} = p'_{d+1}$ . Hence  $m > 0$  and the situation is such that  $S$  and  $S'$  “merge” at  $\theta_c$  where (up to symmetry)  $S$  comes on the left of  $S'$ , and they again “split” at  $\theta_{c+m}$  where  $S$  leaves on the right of  $S'$ , thereby “crossing it”. The full definition, though, covers also the boundary cases of crossing certificates for which  $c \in \{0, n\}$  or  $d \in \{0, \ell\}$  (or both), and when  $S$  and  $S'$  may have no triedge in common; those can be easily examined case by case.

**Lemma 17.** *Let  $(G', T)$  be a full trinet of an edge-weighted plane graph  $G$ , and  $u_i, v_i$ ,  $i = 1, 2$ , be four distinct trinodes. Assume that  $S_i$  from  $u_i$  to  $v_i$  are proper  $T$ -sequences. In  $G' \cup T$ , for  $i = 1, 2$ , there exist routes  $\pi_i$  from  $u_i$  to  $v_i$  having the  $T$ -sequence  $S_i$ , such that  $\pi_1, \pi_2$  are non-crossing, if and only if there exists no crossing certificate for  $S_1, S_2$ .*

Suppose we have two proper  $T$ -sequences  $S, S'$  as in Definition 16. Referring to Definition 16, point a), we call the tricells  $\theta_{c+j} = \theta'_{d+j}$  for  $0 \leq j \leq m$  the *central tricells of the crossing certificate*  $(c, d, m)$ .

*Proof.* Let  $L_i$ ,  $i = 1, 2$ , be the sleeves of  $S_i$  in  $(G', T)$ . Assume that  $(c, d, m)$  is a crossing certificate for  $S_1, S_2$ , and let  $R = (\theta_c, \dots, \theta_{c+m})$  be the sequence of the central tricells of this certificate. Let  $K_i \subseteq L_i$ ,  $i = 1, 2$ , be the plane subgraphs consisting of the copies of the tricells from  $R$  in the sleeve  $L_i$ . Note that  $R$  may repeat the same tricell several times, but in  $L_i$  we have got independent copies of the possibly repeated tricells. We may also assume that  $K_1 = K_2$  since they are both made of copies of the same sequence  $R$  of tricells.

In the above view, Definition 16 says that  $(c, d, m)$  is a crossing certificate iff the elements  $p_c, p'_d, p_{c+m+1}, p'_{d+m+1}$  appear on the outer face of  $K_1 = K_2$  in this cyclic order. Hence, by Jordan’s curve theorem, if there is a crossing certificate for  $S_1, S_2$ , then  $\pi_1, \pi_2$  cannot be non-crossing.

Conversely, we show how to build non-crossing  $\pi_1, \pi_2$  if there is no crossing certificate for  $S_1, S_2$ . For each tricell  $\theta$  of  $T$ , bounded by triedges  $q_1, q_2, q_3$ , we choose arbitrary three edges  $e_j$  from  $q_j$ ,  $j = 1, 2, 3$ , and arbitrary three internally disjoint dual paths  $\sigma_{1,2}, \sigma_{1,3}, \sigma_{2,3}$  contained in  $\theta$  such that  $\sigma_{i,j}$  is a dual path in  $G' \cup T$  connecting the face incident with  $e_i$  to the face incident with  $e_j$ . Furthermore, we denote by  $x_j$  the trinode of  $\theta$  opposite to  $q_j$  and we choose another three arbitrary dual paths  $\rho_1, \rho_2, \rho_3$  contained in  $\theta$  such that  $\rho_j$  is a dual path in  $G' \cup T$  connecting a face incident with  $x_j$  to the face incident with  $e_j$ . We call chosen  $\sigma_{1,2}, \sigma_{1,3}, \sigma_{2,3}, \rho_1, \rho_2, \rho_3$  the representative dual paths of the tricell  $\theta$ .

For the proper  $T$ -sequence  $S_i$ ,  $i = 1, 2$ , we simply compose the route  $\pi_i$  of the appropriate representative dual paths of the tricells determined by  $S_i$ . It is routine to verify that these  $\pi_1, \pi_2$  are non-crossing, if and only if there exists no crossing certificate for  $S_1, S_2$ .  $\square$

### 3.4 Summary of the algorithm

We are now ready to put all of the above results together, in order to summarize the overall algorithm to solve r-MEI, see Algorithm 1. Based thereon, together with Lemmas 12, 14, Corollary 15, and Lemma 17 we obtain:

**Theorem 18.** *Let  $G$  be a connected plane graph with edge weights  $w: E(G) \rightarrow \mathbb{N}_+ \cup \{\infty\}$ , and  $F$  a set of new edges (vertex pairs, in fact) such that  $w(f) = 1$  for all  $f \in F$ . Algorithm 1 finds an optimal solution to  $w$ -weighted r-MEI( $G, F$ ), if a finite solution exists, in time  $\mathcal{O}(2^{\text{poly}(|F|)} \cdot |V(G)|)$ .*

**Input:** a plane graph  $G$ , edge weights  $w: E(G) \rightarrow \mathbb{N}_+ \cup \{\infty\}$ , new edge set  $F$  s.t.  $w(f) = 1$  for  $f \in F$ .  
**Output:** an optimal solution to  $(w$ -weighted) r-MEI( $G, F$ ).

- (1) Compute a full trinet  $(G', T)$ ,  $N(T) := V(F)$ , with the shortest-spanning property of  $T$ .
  - (a) Pick any trinode  $x \in N(T)$  and greedily compute globally-shortest triedges (Def. 9) from  $x$  to all other trinode, using a simple shortest path computation.
  - (b) The remaining triedges can be greedily computed as locally-shortest, one after another.
- (2) For each  $f = uv \in F$ :
  - (a) Compute  $\mathcal{S}_f$  as the set of all of its possible proper  $T$ -sequences from  $u$  to  $v$ . The size of  $\mathcal{S}_f$  is bounded due to Lemma 12(b) as  $\mathcal{O}(2^{\text{poly}(|F|)})$ .
  - (b) For each  $S \in \mathcal{S}_f$ , compute a shortest  $u$ - $v$  route  $\pi_S$  in  $G' \cup T$  among those having the  $T$ -sequence  $S$  (where the length function is induced by  $w$ ), using Corollary 15.
- (3) For each possible set  $\mathcal{P} = \{S_f\}_{f \in F}$  with  $S_f \in \mathcal{S}_f$ :
  - (a) Check, for each pair  $f, f' \in F$ , whether there exists a crossing certificate for  $S_f, S_{f'}$  (e.g., using brute force by Def. 16).  
 Let  $X_{\mathcal{P}}$  be the set of pairs  $\{f, f'\}$  for which such a certificate has been found.
  - (b) If any pair  $\{f, f'\} \in X_{\mathcal{P}}$  requires more than a single crossing (which can be found by checking again for two “independent” crossing certificates of  $S_f, S_{f'}$ ), let  $\text{cr}_{\mathcal{P}} := \infty$ .
  - (c) Otherwise, let  $\text{cr}_{\mathcal{P}} := |X_{\mathcal{P}}| + \sum_{f \in F} \text{len}_w(\pi_{S_f})$ , where  $\text{len}_w$  denotes the length function in the geometric dual of  $G' \cup T$  induced by  $w$ .
- (4) Among all  $\mathcal{P}$  considered in (3), pick the one with smallest  $\text{cr}_{\mathcal{P}} < \infty$ . Let this be  $\mathcal{P}^\circ = \{S_f^\circ\}_{f \in F}$ .
- (5) In the plane graph  $G$ , realize each edge  $f \in F$  following its respective route  $\pi_{S_f^\circ}$ , such that the overall resulting weighted number of crossings is  $\text{cr}_{\mathcal{P}^\circ}$ .
  - (a) (By minimality, no  $\pi_{S_f^\circ}$  will be self-intersecting.) Using well-known postprocessing—removing consecutive crossings between  $f, f'$  by re-routing  $f'$  partially along  $f$  or vice versa—allows to avoid multiple crossings in pairs from  $X_{\mathcal{P}}$  and to make remaining pairs from  $F$  crossing-free.

**Algorithm 1:** Algorithm to solve the (weighted) rigid MEI problem.

Before giving the proof, we need a deeper understanding of the concept of non-crossing routes and crossing certificates, and a detailed specification of the step (3)(b) of Algorithm 1.

By adapting the arguments of Lemma 17, one can actually get the following slight strengthening:

**Claim 19.** *Let  $(G', T)$  be a full trinet of an edge-weighted plane graph  $G$ , and  $u_i, v_i$ ,  $i = 1, 2$ , be four distinct trinode. Let  $\pi_i$ ,  $i = 1, 2$ , be a  $u_i$ - $v_i$  route in  $G' \cup T$ . If there exist simple  $u_i$ - $v_i$  arcs  $b_i$ ,  $i = 1, 2$ , following the route  $\pi_i$ , such that  $b_1$  intersects  $b_2$  in exactly one point  $x$  and they properly cross in  $x$ , then there exists a crossing certificate for the  $T$ -sequences of  $\pi_1$  and  $\pi_2$ .  $\square$*

We say that there exist *two independent crossing certificates* for the  $T$ -sequences  $S, S'$  if there are crossing certificates  $(c, d, m)$  and  $(c', d', m')$  for  $S, S'$  (each one up to possible reversal of  $S'$  as in Definition 16), such that the set of central tricells of  $(c, d, m)$  is disjoint from the set of central tricells of  $(c', d', m')$ . The following can then be straightforwardly obtained from Lemma 17:

**Claim 20.** *Let  $(G', T)$  be a full trinet of an edge-weighted plane graph  $G$ , and  $u_i, v_i$ ,  $i = 1, 2$ , be four distinct trinode. Assume that  $S_i$  from  $u_i$  to  $v_i$  are proper  $T$ -sequences. In  $G' \cup T$ , there exist simple arcs  $b_i$  from  $u_i$  to  $v_i$  such that, for  $i = 1, 2$ ,*

- $b_i$  is contained in the alley of a  $u_i$ - $v_i$  route having the  $T$ -sequence  $S_i$ , and
- $b_1$  intersects  $b_2$  in at most one point,

if and only if there exists no two independent crossing certificates for  $S_1, S_2$ .  $\square$

The implementation of step (3)(b), using Claim 20, hence simply checks by brute force for the existence of two independent crossing certificates for  $S_f, S_{f'}$ .

*Proof of Theorem 18.* In this proof, we will use some of the terminology and notation from the proof of Lemma 12, and refer to the notation of Algorithm 1.

Consider arbitrary  $\mathcal{P} = \{S_f\}_{f \in F}$  as in the step (3). The value of  $\text{cr}_{\mathcal{P}}$  computed in the step (3)(c), provided that  $\text{cr}_{\mathcal{P}^0} < \infty$ , is a lower bound on the number of crossings of any feasible solution of  $\text{r-MEI}(G, F)$  such that, for each  $f \in F$ , the  $T$ -sequence of the arc  $f$  is exactly  $S_f$ . This fact follows directly from Lemma 17 (for the part  $|X_{\mathcal{P}}|$ ) and from Lemma 14 (for the part  $\sum_{f \in F} \text{len}_w(\pi_{S_f})$ ).

By Lemma 12, there is an optimal feasible solution  $D$  to  $\text{r-MEI}(G, F)$  such that, for every  $f \in F$ , the arc of  $f$  in  $D$  has its  $T$ -sequence (with respect to the trinet  $T$  from step (1)) equal to some proper  $S_f \in \mathcal{S}_f$  as computed in (2), and the step (3)(b) does not apply to these values by Claim 20. Consequently,  $\text{cr}_{\mathcal{P}^0} \leq \text{r-ins}(G, F)$  by the lower-bound argument from the previous paragraph. Hence if we can prove that the step (5) indeed can compute a drawing  $D^\circ$  of  $G + F$  with  $\text{cr}_{\mathcal{P}^0}$  weighted crossings, provided  $\text{cr}_{\mathcal{P}^0} < \infty$ , then we complete the proof of Theorem 18.

For  $f \in F$ , let  $b_f$  denote a realization of the edge  $f$  as an arc following the route of  $\pi_{S_f^\circ}$  (such that the  $T$ -sequence of  $b_f$  is  $S_f^\circ$ ), before the postprocessing step (5)(a). By the minimality choice in step (4), we can be sure that  $b_f$  does not cross itself: the self-crossing would induce a non-contractible loop with at least one crossing over  $G'$ , but then there exists a  $T$ -sequence  $S'_f \subset S_f^\circ$ —the sequence  $S_f^\circ$  without the triedges forcing the loop. Replacing these two sequences in  $\mathcal{P}^0$  results in a smaller number of crossings on  $f$  while not increasing the number of crossings at any term in the summation considered in step (3)(c).

Let  $D_1$  denote the drawing of  $G + F$  made of  $G$  and  $\{b_f : f \in F\}$ . Observe that  $\text{cr}_{D_1}(E(G), F) = \sum_{f \in F} \text{len}_w(\pi_{S_f^\circ})$ .

Fix some  $f, f' \in F$ ,  $f = uv$  and  $f' = u'v'$ , such that  $b_f, b_{f'}$  cross each other (properly) more than once. Consider the point set  $p := b_f \cup b_{f'}$  with the outer face incident with, say, the trinode  $u$ . If any of the bounded faces of  $p$  contained a trinode (which cannot be  $u$ ) of  $T$ , then we could “split” the  $T$ -sequences  $S_f^\circ$  and  $S_{f'}^\circ$  into  $S_f^1, S_f^2$  and  $S_{f'}^1, S_{f'}^2$ , each, such that for each of the pairs  $S_f^1, S_{f'}^1$  and  $S_f^2, S_{f'}^2$ , there would exist a crossing certificate by Claim 19. This would, in turn, provide two independent crossing certificates for the  $T$ -sequences  $S_f^\circ, S_{f'}^\circ \in \mathcal{P}^0$  and by the step (3)(b), it would contradict  $\text{cr}_{\mathcal{P}^0} < \infty$ .

Consequently, all the bounded faces of  $p = b_f \cup b_{f'}$  are free of trinodes of  $T$ . This, in particular, means that if we construct  $b_f^1$  from  $b_f$  by re-routing it along a section of  $b_{f'}$  between two consecutive shared points of  $b_f \cap b_{f'}$ , then the  $T$ -sequence of  $b_f^1$  would again be  $S_f^\circ$ . Moreover, since  $b_f, b_{f'}$  have been chosen from their respective shortest routes, the  $t$ -weight of  $b_f^1$  would be equal to  $\text{len}_w(\pi_{S_f^\circ})$  (which is the  $t$ -weight of original  $b_f$ ). Iterating this process, we arrive at a drawing  $D_2$  of  $G + F$  satisfying the following:

- no two edges of  $F$  in  $D_2$  cross more than once,
- $\text{cr}_{D_2}(E(G), F) \leq \text{cr}_{D_1}(E(G), F)$ .

What remains is to observe that two edges  $f, f' \in F$  properly cross each other in  $D_2$  only if  $\{f, f'\} \in X_{\mathcal{P}^0}$ . Indeed, if  $f, f'$  properly cross in  $D_2$ , then this crossing is the only one and there exists a crossing certificate for  $\pi_{S_f^\circ}, \pi_{S_{f'}^\circ}$  by Claim 19, and then  $\{f, f'\} \in X_{\mathcal{P}^0}$  due to the step (3)(a).



To summarize, we get

$$\begin{aligned} \text{cr}_{\mathcal{P}^\circ} &\leq \text{r-ins}(G, F) \leq \text{cr}(D_2) = \text{cr}_{D_2}(E(G), F) + \text{cr}_{D_2}(F, F) \\ &\leq \text{cr}_{D_1}(E(G), F) + |X_{\mathcal{P}^\circ}| = \text{cr}_{\mathcal{P}^\circ} \end{aligned}$$

which proves optimality of the solution computed by Algorithm 1.

Finally, we discuss the runtime bound of Algorithm 1. Let  $k = |F|$ . Step (1) is performed in time  $\mathcal{O}(k^2 \cdot |V(G)|)$  using  $3(2k) - 6$  calls to a linear shortest path algorithm. Step (2) takes time  $\mathcal{O}(k \cdot 2^{\text{poly}(k)} \cdot k|V(G)|)$  by Corollary 15. Step (3) is iterated  $\mathcal{O}(2^{\text{poly}(k) \cdot k})$  times, and each iteration takes time polynomial in  $k$  (independently of  $G$ ) even by brute force. Step (4) takes only time  $\mathcal{O}(2^{\text{poly}(k) \cdot k})$ . Finally, step (5) performs  $k$  computations in  $\mathcal{O}(|V(G)|)$ , to realize each  $f \in F$  in  $G$ , and then a number of concurrent re-routings which can be bounded by an amortized analysis: every of the  $k$  routes is of length  $\mathcal{O}(|V(G)|)$  and each element of it could be re-routed at most once towards each of the  $k - 1$  remaining routes, summing to  $\mathcal{O}(k^2 \cdot |V(G)|)$ .

The above analysis sums up to overall  $\mathcal{O}(2^{\text{poly}(k)} \cdot |V(G)|)$  time.  $\square$

## 4 General MEI

Now, we may turn our attention to the general MEI problem, where the embedding of the planar graph  $G$  is not prespecified. See also the appendix for details. Recall that triconnected planar graphs have a unique embedding (up to mirroring), but already biconnected graphs have an exponential number of embeddings in general. As it is commonly done in insertion problem since [20], we will use the *SPR-tree* datastructure (sometimes also known as SPQR-tree) to encode and work with all these possible embeddings. It was first defined in slightly different form in [15], based on prior work of [2, 32]. It can be constructed in linear time [19, 25] and only requires linear space.

**Definition 21** (SPR-tree, cf. [7]). Let  $G$  be a biconnected graph with at least three vertices. The *SPR-tree*  $\mathcal{T}$  of  $G$  is the unique smallest tree satisfying the following properties:

- i) Each node  $\nu$  in  $\mathcal{T}$  holds a specific (small) graph  $S_\nu = (V_\nu, E_\nu)$ , with  $V_\nu \subseteq V(G)$ , called a *skeleton*. Each edge  $e$  of  $E_\nu$  is either a *real* edge  $e \in E(G)$ , or a *virtual* edge  $e = xy \notin E(G)$  (while still,  $x, y \in V(G)$ ).
- ii)  $\mathcal{T}$  has three different node types with the following skeleton structures: **(S)**  $S_\nu$  is a simple cycle; **(P)**  $S_\nu$  consists of two vertices and at least three multiple edges between them; **(R)**  $S_\nu$  is a simple triconnected graph on at least four vertices.
- iii) For every edge  $\nu\mu$  in  $\mathcal{T}$  we have  $|V_\nu \cap V_\mu| = 2$ . These two common vertices, say  $x, y$ , form a vertex 2-cut (a *split pair*) in  $G$ . Skeleton  $S_\nu$  contains a specific virtual edge  $e_\mu \in E(S_\nu)$  that represents the node  $\mu$  and, symmetrically, some specific  $e_\nu \in E(S_\mu)$  represents  $\nu$ ; both  $e_\nu, e_\mu$  have the ends  $x, y$ .
- iv) The original graph  $G$  can be obtained by recursively applying the following operation of merging: For an edge  $\nu\mu \in E(\mathcal{T})$ , let  $e_\mu, e_\nu$  be the pair of virtual edges as in (iii) connecting the same  $x, y$ . A *merged* graph  $(S_\nu \cup S_\mu) - \{e_\mu, e_\nu\}$  is obtained by gluing the two skeletons together at  $x, y$  and removing  $e_\mu, e_\nu$ .  $\diamond$

The central theorem of [20] states that we can find an optimal embedding to insert a single edge  $uv$  by looking at the shortest path in  $\mathcal{T}$  between a node whose skeleton contains  $u$  and a node whose

skeleton contains  $v$ . For each skeleton along this path, one considers the partial routes between the virtual edge representing  $u$  (or  $u$  itself) and the virtual edge representing  $v$  (or  $v$  itself). In case of S- and P-nodes this route requires no crossings (by choosing a suitable embedding in the latter case); for an R-node  $\nu$ , the route is a shortest path in the dual of its skeleton: if the primal edge is an original edge, the length of its dual edge is the primal edge's weight; if the primal edge is a virtual edge  $xy$ , representing node  $\mu$ , the length of its dual edge is the minimum- $xy$ -cut in  $P_\mu$ , where we  $P_\mu$  is the *pertinent graph* of  $\mu$  arising from merging all skeletons of the subtree rooted at  $\mu$ , minus the edge  $e_\nu$ . By picking *any* embedding of  $P_\mu$  and computing a shortest dual path through it, we can compute this cut size in linear time. See [20] for details.

We consider our SPR-tree  $\mathcal{T}$  of  $G$  rooted at any node, and devise a dynamic programming scheme to solve MEI bottom-up over  $\mathcal{T}$ . We observe that every non-root skeleton  $S_\nu$  contains a virtual edge  $e_\nu$  that represents its father in  $\mathcal{T}$ . Any further virtual edges correspond to children of  $\nu$  in  $\mathcal{T}$ . Since we already know how to solve r-MEI, it shall suffice to describe which r-MEI problems we need to solve at each SPR-tree node  $\nu$  (including the root node), assuming we already solved the corresponding subproblems at their children. The overall MEI solution can then be obtained by selecting a solution in the root with the least number of crossings.

We say a virtual edge in  $S_\nu$  is *dirty* if it contains an end vertex of a new edge  $f \in F$ . Hence, at most  $2k$  edges (a constant number) of  $S_\nu$  are dirty. For R- and S-nodes, we only have to consider their unique (up to mirroring, in case of R) embeddings. A P-node whose skeleton contains  $p$  edges, however, allows  $(p-1)!$  embeddings. Based on the following claim (which can be shown with a straight-forward redrawing argument), we only need to consider up to  $(2k)!$  embeddings for each P-node, which is constant for constant  $k$ .

**Claim 22.** *Let  $\nu$  be a P-node in the SPR-tree  $\mathcal{T}$  of  $G$ . There is an optimal embedding of  $G$  for the MEI problem, where all non-dirty virtual edges are consecutive in the embedding of  $S_\nu$ .*

Let  $S'_\nu$  be a considered embedding of  $S_\nu$ . Consider each virtual edge  $xy$ , representing node  $\mu$  (possibly,  $\mu$  is the father node), in  $S_\nu$ . If  $xy$  is not dirty, we set its weight to the size of the minimum- $xy$ -cut in  $P_\mu$  (as for the single edge insertion case). If  $xy = e$  is dirty, we modify it with the following gadget: Set the weight of  $e$  to  $\infty$ , and add two new *side edges*  $e', e''$  connecting  $x$  and  $y$ . One is directly to the left, the other directly to the right of  $e$ . We will further modify these side edges in the following.

Consider what can happen at the subdrawing of (embedded) component  $P_\mu$  in the context of whole  $G$  when considering any specific new edge  $f \in F$ : (i) if  $f$  has exactly one end in  $P_\mu$ , it will enter the component; (ii) if  $f$  has no end in  $P_\mu$ , it may cross through the component; (iii) if  $f$  has both ends in  $P_\mu$ , it may leave the component and re-enter  $P_\mu$  at another position. Furthermore, and in contrast to the single edge insertion, it may happen that  $f$  (independent of its end points) crosses  $P_\mu$  multiple times. However, since we consider a fixed embedding  $S'_\nu$ , we know from Lemma 12 that the latter number is bounded by a constant, depending only on  $k$ . Hence, there are only a bounded number of enterings/leavings at  $P_\mu$ , and we can simply consider all such possible situations (including all possible orders of the enterings/leavings). For each such situation, we now subdivide the edges  $e'$  and  $e''$  accordingly: chiefly put, if, e.g., we consider the case of an edge  $f = uv$  coming from a vertex  $u \notin V(P_\mu)$  and crossing  $P_\mu$  twice before finally entering it to reach  $v \in V(P_\mu)$ , we generate (for  $f$ ) overall five vertices on  $e', e''$ , say  $v_1, \dots, v_5$ . Within the context of the dynamic programming subproblems at  $P_\mu$ , we then consider (for each embedding of  $S_\mu$ ) the r-MEI problem w.r.t. *subedges*  $v_1v_2, v_3v_4, v_5v$ . Overall, we have to store the best solution (over all embeddings of  $S_\mu$ ) for each r-MEI problem constructed of all such subedges, for each edge  $f \in F$ , each possible number of crossings of  $f$  through  $P_\mu$ , each possible assignment of thereby induced subdivision vertices to  $e'$  or  $e''$ , and all possible orders at  $e'$  and  $e''$ . Within a subproblem at  $\nu$ , we then

consider the edge  $uv_1$  instead of  $f$  (in fact, this edge may be further split into several subedges due to further dirty virtual edges in  $S_\nu$  considered to be crossed by  $f$ , and/or if  $u$  is contained in a pertinent graph of another virtual edge).

Hence, we only need to store a constant (bounded by a function in  $k$ ) number of solutions at each SPR-tree node. Each solution can be obtained using the above algorithm for r-MEI in  $\mathcal{O}(|V(G)|)$  time, and there are at most  $\mathcal{O}(|V(G)|)$  SPR-tree nodes. Instead of the naïve quadratic runtime bound, we even achieve a linear runtime bound by observing that the union of all skeletons is still only of linear size. We obtain, as given in the introduction:

**Theorem 23** (The biconnected case of Theorem 1). *Let  $G$  be a planar biconnected graph on  $n$  vertices, and  $F$  a set of  $k$  new edges (vertex pairs, in fact) where  $k$  is a constant. We can solve  $\text{MEI}(G, F)$  in  $\mathcal{O}(n)$  time.*

For essentially all known insertion algorithms (in particular single edge insertion [20], vertex insertion [9], and MEI approximation [10]), one can typically first describe the case of biconnected graphs (using SPR-trees). Then, it is relatively straight-forward to lift the algorithms to connected graphs, by considering BC-trees (see below). Interestingly, this seems much more complicated in case of exact MEI:

Consider the well-known block-cut tree (BC-tree) to decompose any connected graph into its blocks (biconnected components). Using analogous techniques as in [10], we extend our dynamic programming approach by amalgamating the BC-tree with the blocks' respective SPR-trees, to obtain a linear-sized *con-tree*, with an additional node type **C**, for cut vertices. In our bottom-up approach, at a cut vertex  $c$ , we need to consider all possibilities to “glue” the  $c$ -incident dirty blocks (blocks with at least one end of some edge  $f \in F$ ) together. However, we cannot easily bound this number by a function purely in  $k$ : we not only have to consider all orders of these blocks, but also all possible nestings, which introduces a dependency on  $\Delta_{\text{cut}}$ , the maximum degree of the cut vertices in  $G$ . Hence, for only connected  $G$ , we obtain the slightly weaker result:

**Theorem 24** (The connected case of Theorem 1). *Let  $G$  be a planar connected graph on  $n$  vertices, and  $F$  a set of  $k$  new edges (vertex pairs, in fact), where  $k$  and the maximum degree of the cut vertices in  $G$  are constant. We can solve  $\text{MEI}(G, F)$  in  $\mathcal{O}(n)$  time.*

As sketched in the main body of the paper, we first develop a dynamic programming algorithm over the SPR-tree decomposition  $\mathcal{T}$  of planar  $G$ . This algorithm considers *dirty nodes* bottom-up; a decomposition node  $\nu$  is *dirty* if its pertinent graph contains at least one vertex incident to  $F$ . Observe that if a node is dirty, so is its parent. The root node (whose pertinent graph we may define as  $G$  itself) is always dirty.

**Subproblems at non-root nodes.** We start with formally defining the subproblems to be solved and stored at each dirty non-root decomposition node. Let  $\nu$  be such a node and let  $e = xy \in E(S_\nu)$  be the virtual edge in the skeleton of  $\nu$  corresponding to its parent node  $\varrho$ . Recall that the pertinent graph  $P_\nu$  arises from  $S_\nu$  by merging the skeletons of the subtree rooted at  $\nu$  and removing the sole remaining virtual edge ( $e$ ). We consider the 3-partition of  $F$  into  $F_0, F_1, F_2$ , where  $F_0$  are the edges without an end in  $V(P_\nu) \setminus \{x, y\}$ ,  $F_1$  are the edges with one end in  $V(P_\nu) \setminus \{x, y\}$  and the other not in  $V(P_\nu)$ , and  $F_2$  are the edges with one end in  $V(P_\nu) \setminus \{x, y\}$  and the other in  $V(P_\nu)$ .

By definition, the graph  $P_+ := P_\nu + e$  is planar, and  $e$  represents the “rest of the graph” disjoint from  $P_\nu$ . We are, intuitively, interested in the best embedding  $P_+^o$  of  $P_+$  to

- (a) route the edges of  $F_1$  from a side of  $e$  to its end in  $V(P_\nu) \setminus \{x, y\}$ ; observe that we may care from which side of  $e$  the new edge emanates.

But these are not the only routes to consider in an optimal solution:

- (b) edges  $uv \in F_2$  may be routed completely within  $P_\nu$ , or go from  $u$  to some side of  $e$  (into the “rest of the graph”), and from some side  $e$  (from the “rest of the graph”; either the same or the other side) to  $v$ ;
- (c) any edge of  $F$  may be routed through  $P_\nu$ , i.e., from one side of  $e$  to the other side, without crossing  $e$ .

Formally, we can define a *routing query* as a pair  $(s, t)$ , where  $s$  and  $t$  are each either referencing a specific side of  $e$  or a vertex in  $V(P_\nu) \setminus \{x, y\}$ . We will use  $e', e''$  to denote the two different sides of  $e$ . In such a routing query, we ask for a routing of a new edge between  $s$  and  $t$  in  $P_+$ , without crossing over  $e$ .

Lemma 12 (which holds for every fixed embedding, and hence for each possible embedding) showed that a triedge of trinet  $T$  is crossed at most  $8k^4 = \mathcal{O}(\text{poly}(k))$  times. When computing a shortest route (w.r.t. some  $T$ -sequence) between two succeeding triedges, we clearly have the property that any edge within the corresponding tricell is crossed at most once. Hence:

**Corollary 25.** *In an optimal solution to  $\text{MEI}(G, F)$ , each edge  $f \in F$  crosses any edge  $e \in G$  at most  $\xi := \text{poly}(k)$  times.*

Since this corollary also holds for virtual edges in a skeleton, we have the same upper bound for crossings through a two-connected component  $P_\nu$ .

In our dynamic programming scheme, we will hence—for each possible set of routing queries—store the minimum number of crossings necessary over all embeddings of  $P_+$ . A specific set of routing queries (to be described in details below) is hence a *subproblem*, and the corresponding number of crossings (together with the embedding of  $P_+$  and the corresponding routings, if desired) is a *subsolution*. It remains to discuss the number of subproblems for  $\nu$ .

**Lemma 26.** *Each subproblem specifies at most  $\mathcal{O}(\text{poly}(k))$  routing queries. The total number of subproblems to consider at any node  $\nu$  is bounded by  $\mathcal{O}(\text{poly}(k)!)$ .*

*Proof.* Consider the routing types (a)–(c) as above:

- (a) For each edge  $f = uv \in F_1$  with  $u \in V(P_\nu) \setminus \{x, y\}$ , we have to pick one of the two routing queries  $(e', u)$ ,  $(e'', u)$ .
- (b) For each edge  $f = uv \in F_2$ , we have to pick one out of five options: (i) a single routing query  $(u, v)$ ; (ii)–(v) two routing queries  $(u, e^{(1)})$ ,  $(e^{(2)}, v)$ , with  $e^{(1)}, e^{(2)} \in \{e', e''\}$ .
- (c) Finally, for each  $f \in F$ —except for those  $F_2$ -edges that picked option (i)—we have additional up to  $\xi$  routing queries. Each such additional query is of one of four types:  $(e^{(1)}, e^{(2)})$ , with  $e^{(1)}, e^{(2)} \in \{e', e''\}$ .

Overall, this gives up to  $r := |F_1| + 2|F_2| + \xi|F| = \mathcal{O}(\text{poly}(k))$  routing queries.

The number of choices for such a set of routing queries is at most  $2^{|F_1|} \cdot 5^{|F_2|} \cdot (4^\xi)^{|F|} = \mathcal{O}(5^{\text{poly}(k)})$ . However, up to now we did not consider a crucial interplay of these individual routing queries: We need to take all possible orderings of the edges emanating from a side of  $e$  into account: Sides of  $e$  arise at most  $2r$  times over all queries, and we hence have at most  $(2r)!$  orderings to consider. Thus, we overall obtain  $\mathcal{O}(5^{\text{poly}(k)} \cdot \text{poly}(k)!) = \mathcal{O}(\text{poly}(k)!)$  subproblems.  $\square$

**Dynamic programming and root node.** Finally, we have to describe how to use these subproblems to efficiently compute MEI. The validity of this approach for non-dirty pertinent graphs was already established in [20]. As mentioned, we consider dirty nodes bottom-up.

Let  $\nu$  be the considered SPR-tree node with skeleton  $S_\nu$ . Let  $e_\rho \in E(S_\nu)$  be the virtual edge corresponding to  $\nu$ 's father  $\rho$  (if it exists), and  $e_1, \dots, e_\ell$  ( $e'_1, \dots, e'_{\ell'}$ ) the dirty (non-dirty) virtual edges in  $S_\nu$  corresponding to the children  $\mu_1, \dots, \mu_\ell$  ( $\mu'_1, \dots, \mu'_{\ell'}$ , respectively). We need to show that we can solve each subproblem at  $\nu$  purely using  $S_\nu$  and the solutions to the subproblems of the dirty children. In particular, we may not expand the skeleton to the pertinent graph (for which the  $\nu$ -subproblems are actually defined).

*Subproblems, embeddings, and the root.* Assume,  $\nu$  is a non-root node, then we have to solve  $\chi := \mathcal{O}(\text{poly}(k)!)^2$  many subproblems. For each subproblem, we are given a set of  $r := \mathcal{O}(\text{poly}(k))$  routing queries, and want to find the optimal solution over all embeddings of  $P_+$ . As a first step, we recall that there are only a bounded number of embeddings for  $S_\nu$  (1, 2, and  $(2k)!$  in case of an S-, R-, and P-node; let  $\chi' = \mathcal{O}((2k)!)$ ), and we may hence enumerate each one explicitly.

The routing queries of the considered subproblem give rise to the following gadget: Set the weight of  $e_\rho$  to  $\infty$ , and introduce two edges  $e'$  and  $e''$  parallel to  $e_\rho$ , one directly to its left, one directly to its right. Now subdivide these to edges such that there is a vertex on  $e'$  ( $e''$ ) if a routing query specifies the edge side  $e'$  ( $e''$ ). Furthermore, these vertices are ordered according to the specification of the subproblem. Let  $S'_\nu$  denote the embedded graph arising from this construction; we do not consider  $e_\rho$  as a virtual edge in the following any more. Instead of considering the original new edge set  $F$ , we will now consider the routing queries  $(s, t)$  as edges  $st \in F'$  (a new set  $F'$ ) to be inserted.

If  $\nu$  is the root node, we also have to consider all its skeleton's possible embeddings  $S'_\nu$  individually, but there is no specific subproblem to consider and we simply set  $F' := F$  without any gadget construction. From now on, we do the same steps, independent on whether considering the root node, or a specific subproblem at a non-root node.

*Virtual edges.* For each non-dirty virtual edge  $e'_i = ab$ ,  $1 \leq i \leq \ell'$ , we set the weight of  $e'_i$  to the minimum- $ab$ -cut in the pertinent graph of  $\mu'_i$ . Note that these values can be constructed bottom up in overall linear time as a preprocessing.

Now, for each dirty virtual edge  $e_i$ ,  $1 \leq i \leq \ell$  ( $\ell \leq 2k$ ), we construct a gadget analogous to the gadget for  $e_\rho$ : Edge  $e_i$  gets weight  $\infty$ , we add two edges  $e'_i, e''_i$  left and right of  $e_i$ , and subdivide them according to a subproblem at  $\mu_i$ .—To do this, we have to enumerate all possible choices of subproblems at all virtual edges. So this construction yields  $\chi'' := \mathcal{O}((\text{poly}(k)!)^{2k})$  different choices, each of which we consider individually. Observe: If, for some edge  $f' \in F'$  that resides within some  $P_{\mu_j}$ , we chose a routing query of type (b)(i), we do not consider subsolutions at any virtual edge where there are type (c) queries w.r.t.  $f'$ . We call such an edge  $f'$  a *suppressed edge*.

We denote the so-modified plane graph by  $S''_\mu$ , and now have to decide what happens to our new edges  $F'$ . Each edge in  $F'$  corresponds to some edge in  $F$ . Furthermore, we add each non-suppressed edge  $f \in F$  to  $F'$  if it has no corresponding edge in  $F'$ . We observe that for each vertex  $w \in P_\nu \setminus S_\nu$  that is an end in  $F'$ , there is a unique *replacement vertex*  $r(w)$  in  $S_\nu$ —it arises from a query  $(r, r(w))$  (unoriented) within a subproblem at some dirty virtual edge.

For each original edge  $f = uv \in F$ , we hence get a partial order of routing queries corresponding to it: either  $u$  ( $v$ ) or its replacement vertex  $r(u)$  ( $r(v)$ , respectively) is in  $S'_\nu$ , so we start (end) there. There may or may not be a routing query starting at  $u$  (ending at  $v$ ), which we would update to use  $r(u)$  ( $r(v)$ ) instead of  $u$  ( $v$ ). Now, between this start and end,  $f$  may have to “visit” former queries of type (c) (whose ends are now represented by subdivision vertices at edges  $e'_j, e''_j$ ,  $1 \leq j \leq \ell$ ). While these former queries are totally ordered for each individual dirty virtual edge, it is unclear in

which order  $f$  visits the different virtual edges. We will enumerate all possible orders to visit each of the  $\mathcal{O}(k)$  dirty virtual edges up to  $\mathcal{O}(\xi)$  times; there are hence  $\chi''' := \mathcal{O}\left(\frac{(k\xi)!}{k\xi!}\right) = \mathcal{O}(\text{poly}(k)!)!$  different visit orderings for each edge of  $f \in F$ . Every visit order induces an unambiguous set  $T_f$  of (new) routing queries to draw part of  $f$  within  $S_\nu$ : from  $f$ 's start to the vertex representing the beginning of a former query, from the vertex representing the end of the last former query to the beginning of the next former query, and so on, until finally from the vertex representing the end of the last former query to  $f$ 's end. Such a set  $T_f$  hence has size at most  $\mathcal{O}(k\xi) = \mathcal{O}(\text{poly}(k))$ .

So, finally, we obtain an instance  $\text{r-MEI}(S'_\nu, F'')$ , where  $|V(S'_\nu)| = \mathcal{O}(|S_\nu| \cdot k\xi) = \mathcal{O}(|S_\nu| \cdot \text{poly}(k))$  and  $F''$  is the set of all routing queries (interpreted as unordered new edges) obtained from  $F'$  by considering each  $T_f$  (for all  $f \in F$ ). We have  $|F''| = \mathcal{O}(k \text{poly}(k)) = \mathcal{O}(\text{poly}(k))$ . The total cost of the considered subsolution (and also for the solution at the root node) is the minimum number of crossings over all possible  $\text{r-MEI}$  instances constructed as above *plus* the numbers of crossings given by the corresponding individual subsolutions realized at the dirty virtual edges. We have:

**Lemma 27.** *We settle the root node—and any specific subproblem at a non-root node—with  $\mathcal{O}(\chi' \cdot \chi'' \cdot \chi'''^k)$  calls to  $\text{r-MEI}$ . We settle each dirty non-root node with  $\mathcal{O}(\chi \cdot \chi' \cdot \chi'' \cdot \chi'''^k)$  calls to  $\text{r-MEI}$ .*

**Theorem 28** (Detailed version of Theorem 23). *Let  $G$  be a planar biconnected graph on  $n$  vertices, and  $F$  a set of  $k$  new edges (vertex pairs, in fact) where  $k$  is a constant. We can solve  $\text{MEI}(G, F)$  in  $\mathcal{O}(n \cdot (\text{poly}(k)!)^{\Theta(k)}) = \mathcal{O}(n) \cdot k^{k^{\mathcal{O}(1)}}$  time.*

*Proof.* First, due to Theorem 18, each individual  $\text{r-MEI}$  instance in our setting can be computed within  $\mathcal{O}(|V(S_\nu)| \cdot \text{poly}(k) \cdot 2^{\text{poly}(k)}) = \mathcal{O}(|V(S_\nu)| \cdot 2^{\text{poly}(k)})$  time. Furthermore, the union over all SPR-tree skeletons has still linear size  $\mathcal{O}(n)$ . We hence obtain the overall runtime

$$\begin{aligned} \mathcal{O}(\chi \cdot \chi' \cdot \chi'' \cdot \chi'''^k \cdot n \cdot 2^{\text{poly}(k)}) &= \mathcal{O}(n \cdot \text{poly}(k)! \cdot (2k)! \cdot (\text{poly}(k)!)^{2k} \cdot (\text{poly}(k)!)^k \cdot 2^{\text{poly}(k)}) \\ &= \mathcal{O}(n \cdot (\text{poly}(k)!)^{\Theta(k)}) \end{aligned}$$

□

**Connected Case.** Until now, we only considered biconnected  $G$ . In case of only connected  $G$ , we can first decompose (in linear time)  $G$  into its biconnected components (blocks), and establish a BC-tree  $\mathcal{B}$ . This tree has two types of nodes: For each block of  $G$ , we have a node of type **(B)**; for each cut vertex in  $G$ , we have a node of type **(C)**. We have an edge  $\beta\gamma$  in  $\mathcal{B}$  if, and only if,  $\beta$  is a B-node,  $\gamma$  is a C-node, and the block of  $\beta$  contains the cut vertex of  $\gamma$ . We may root  $\mathcal{B}$  arbitrarily at any dirty block; we say a block is *dirty* if it contains at least one end of  $F$  (other than possibly its parent cut vertex). Clearly, we can iteratively prune non-dirty B-leaves.

Now, we can construct a combined tree  $\mathcal{C}$ : For each block  $B$  in  $G$ , we construct (and root) its SPR-tree  $\mathcal{T}_B$ . In  $\mathcal{B}$ , we replace each B-node with the root vertex of the block's corresponding SPR-tree. Now, we can run the dynamic programming algorithm over  $\mathcal{C}$  instead of a single SPR-tree.

Let  $\nu$  be a non-C-node whose parent is a C-node  $\gamma$  corresponding to cut vertex  $c \in S_\nu$ . We need to redefine the subproblems to consider at  $\nu$ : instead of considering routing queries that attach to one of the two sides of the parent virtual edge, our routing queries may now attach to  $c$  in a specified order and through specified faces incident to  $c$ . We therefore introduce the gadget—for each considered embedding  $S'_\nu$  of  $S_\nu$ —obtained by planarly replacing  $c$  by a simple cycle  $C$ . The  $c$ -incident edges are attached to  $C$  such that the contraction of  $C$  again gives  $S'_\nu$ . When considering the routing queries, instead of the two choices of the side of the parent virtual edge, we now hence

have a  $\delta(c) = \mathcal{O}(\Delta_{\text{cut}})$ -fold choice over the segment of  $C$  where to attach to, where  $\Delta_{\text{cut}}$  denotes the maximum degree over all cut vertices.

In our dynamic programming, we will perform no operation at C-nodes, but let  $\nu$  now be a node with a C-child  $\gamma$  corresponding to cut vertex  $c \in S_\nu$ . Analogous to above—in each considered embedding  $S'_\nu$  of  $S_\nu$ —we planarly replace  $c$  by a cycles  $C$ . On  $C$ , we realize all subsolutions of all (at most  $2k$ ) children of  $\gamma$ , in all possible combinations. Except for these modifications, the algorithm remains unchanged, and we obtain:

**Theorem 29** (Detailed version of Theorem 24). *Let  $G$  be a planar connected graph on  $n$  vertices, and  $F$  a set of  $k$  new edges (vertex pairs, in fact), where  $k$  and  $\Delta_{\text{cut}}$ —the maximum degree of the cut vertices in  $G$ —are constant. We can solve  $\text{MEI}(G, F)$  in  $\mathcal{O}(n \cdot \Delta_{\text{cut}}^{\Theta(k)} \cdot (\text{poly}(k)!)^{\Theta(k)})$  time.*

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## References

- [1] S. N. Bhatt and F. T. Leighton. A framework for solving vlsi graph layout problems. *J. Comput. Syst. Sci.*, 28(2):300–343, 1984.
- [2] D. Bienstock and C. L. Monma. On the complexity of embedding planar graphs to minimize certain distance measures. *Algorithmica*, 5(1):93–109, 1990.
- [3] S. Cabello. Hardness of approximation for crossing number. *Discrete & Computational Geometry*, 49(2):348–358, 2013.
- [4] S. Cabello and B. Mohar. Crossing number and weighted crossing number of near-planar graphs. *Algorithmica*, 60(3):484–504, 2011.
- [5] S. Cabello and B. Mohar. Adding one edge to planar graphs makes crossing number and 1-planarity hard. *SIAM J. Comput.*, 42(5):1803–1829, 2013.
- [6] B. Chazelle. A theorem on polygon cutting with applications. In *23rd Annual Symposium on Foundations of Computer Science, Chicago, Illinois, USA, 3-5 November 1982*, pages 339–349. IEEE Computer Society, 1982.
- [7] M. Chimani. *Computing Crossing Numbers*. PhD thesis, TU Dortmund, Germany, 2008. Online, e.g., at [www.cs.uos.de/theoinf](http://www.cs.uos.de/theoinf).
- [8] M. Chimani and C. Gutwenger. Advances in the planarization method: Effective multiple edge insertions. *J. Graph Algorithms Appl.*, 16(3):729–757, 2012.
- [9] M. Chimani, C. Gutwenger, P. Mutzel, and C. Wolf. Inserting a vertex into a planar graph. In *Proc. SODA '09*, pages 375–383, 2009.
- [10] M. Chimani and P. Hliněný. A tighter insertion-based approximation of the crossing number. In *Proc. ICALP '11*, volume 6755 of *LNCS*, pages 122–134. Springer, 2011.
- [11] M. Chimani, P. Hliněný, and P. Mutzel. Vertex insertion approximates the crossing number for apex graphs. *European Journal of Combinatorics*, 33:326–335, 2012.
- [12] J. Chuzhoy. An algorithm for the graph crossing number problem. In *Proc. STOC '11*, pages 303–312. ACM, 2011.

- [13] J. Chuzhoy, Y. Makarychev, and A. Sidiropoulos. On graph crossing number and edge planarization. In *Proc. SODA '11*, pages 1050–1069. ACM Press, 2011.
- [14] É. C. de Verdière and A. Schrijver. Shortest vertex-disjoint two-face paths in planar graphs. *ACM Transactions on Algorithms*, 7(2):19, 2011.
- [15] G. Di Battista and R. Tamassia. On-line planarity testing. *SIAM Journal on Computing*, 25:956–997, 1996.
- [16] G. Even, S. Guha, and B. Schieber. Improved approximations of crossings in graph drawings and VLSI layout areas. *SIAM J. Comput.*, 32(1):231–252, 2002.
- [17] I. Gitler, P. Hliněný, J. Leanos, and G. Salazar. The crossing number of a projective graph is quadratic in the face-width. *Electronic Notes in Discrete Mathematics*, 29:219–223, 2007.
- [18] M. Grohe. Computing crossing numbers in quadratic time. *J. Comput. Syst. Sci.*, 68(2):285–302, 2004.
- [19] C. Gutwenger and P. Mutzel. A linear time implementation of SPQR trees. In *Proc. GD '00*, volume 1984 of *LNCS*, pages 77–90. Springer, 2001.
- [20] C. Gutwenger, P. Mutzel, and R. Weiskircher. Inserting an edge into a planar graph. *Algorithmica*, 41(4):289–308, 2005.
- [21] J. Hershberger and J. Snoeyink. Computing minimum length paths of a given homotopy class. *Comput. Geom.*, 4:63–97, 1994.
- [22] P. Hliněný and M. Chimani. Approximating the crossing number of graphs embeddable in any orientable surface. In *Proc. SODA '10*, pages 918–927, 2010.
- [23] P. Hliněný and G. Salazar. On the crossing number of almost planar graphs. In *Proc. GD '05*, volume 4372 of *LNCS*, pages 162–173. Springer, 2006.
- [24] P. Hliněný and G. Salazar. Approximating the crossing number of toroidal graphs. In *Proc. ISAAC '07*, volume 4835 of *LNCS*, pages 148–159. Springer, 2007.
- [25] J. E. Hopcroft and R. E. Tarjan. Dividing a graph into triconnected components. *SIAM Journal on Computing*, 2(3):135–158, 1973.
- [26] K.-I. Kawarabayashi and B. Reed. Computing crossing number in linear time. In *Proc. STOC 2007*, pages 382–390, 2007.
- [27] P. Klein, S. Rao, M. Rauch, and S. Subramanian. Faster shortest-path algorithms for planar graphs. In *STOC 94*, pages 27–37, 1994.
- [28] Y. Kobayashi and C. Sommer. On shortest disjoint paths in planar graphs. *Discrete Optimization*, 7(4):234–245, 2010.
- [29] D. Lee and F. P. Preparata. Euclidean shortest paths in the presence of rectilinear barriers. *Networks*, 14(3):393–410, 1984.
- [30] M. Schaefer. The graph crossing number and its variants: A survey. *Electronic Journal of Combinatorics*, #DS21, May 15, 2014.



- [31] M. Thorup. Undirected single source shortest paths with positive integer weights in linear time. *Journal of the ACM*, 46:362–394, 1999.
- [32] W. T. Tutte. *Connectivity in graphs*, volume 15 of *Mathematical Expositions*. University of Toronto Press, 1966.
- [33] T. Ziegler. *Crossing Minimization in Automatic Graph Drawing*. PhD thesis, Saarland University, Germany, 2001.